A NEW TEXT BOOK
OF
ANALYTICAL SOLID GEOMETRY
(Volume I)

A NEW TEXT BOOK

OF

ANALYTICAL SOLID GEOMETRY

[Volume I]

By

S. M. MATHUR

Department of Mathematics

UNIVERSITY OF JODHPUR, JODHPUR

OXFORD & IBH PUBLISHING CO.

CALCUTTA BOMBAY NEW DELHI

Published by Oxford & I B H Publishing Co., Oxford Building, N-88, Connaught Circus, New Delhi-1 and printed by V. P. Dhawan at R. K. Printers, 80-D Kamla Nagar, Delhi-7

Preface

The present little volume is not a treatise, but is designed as an introductory text book that will serve the purpose of the students appearing at B.Sc., B. Engineering and M.Sc. examinations of all Indian and foreign universities. This volume is meant to open doors for the student and to give him understanding and preparation which will help him to proceed further to the new frontiers of knowledge, carrying a fundamental background of the classical Analytical Solid Geometry. It is the outgrowth of a course taught by me to the undergraduate and postgraduate students over the last several years. On the basis of my many years of experience as a teacher of this subject, I have tried to present it in a manner most useful to the student community all over the world. The theory has been developed through the powerful tool of vectors. Side by side classical treatment is also given. The chapters have been divided into various sections corresponding to particular principles. At the end of every section, an example is added which illustrates the application of the principles given in the section. At the end of every chapter, a miscellaneous revision example is appended, which serves as a complete review of the material of that chapter. Alternative methods have been provided at several places. 'Aid to memory' is provided at some places for assimilation purpose. Most of these problems have been drawn from the examination papers of various universities and the standard works on the subject. I sincerely acknowledge my indebtedness to these sources.

I take this opportunity to pay my sincere thanks to my fellow friends here and elsewhere who have kept in me the flagrant desire of presenting the subject matter in such a novel fashion. I also thank the director of Oxford and IBH Calcutta for his zeal and enthusiasm

in the production of such a matchless book. Lastly, I also thank the press staff for their efforts in producing this book so nicely.

Constructive suggestions for the improvement of this book will be highly appreciated.

University of Jodhpur ; S. M. MATHUR

January, 1969.

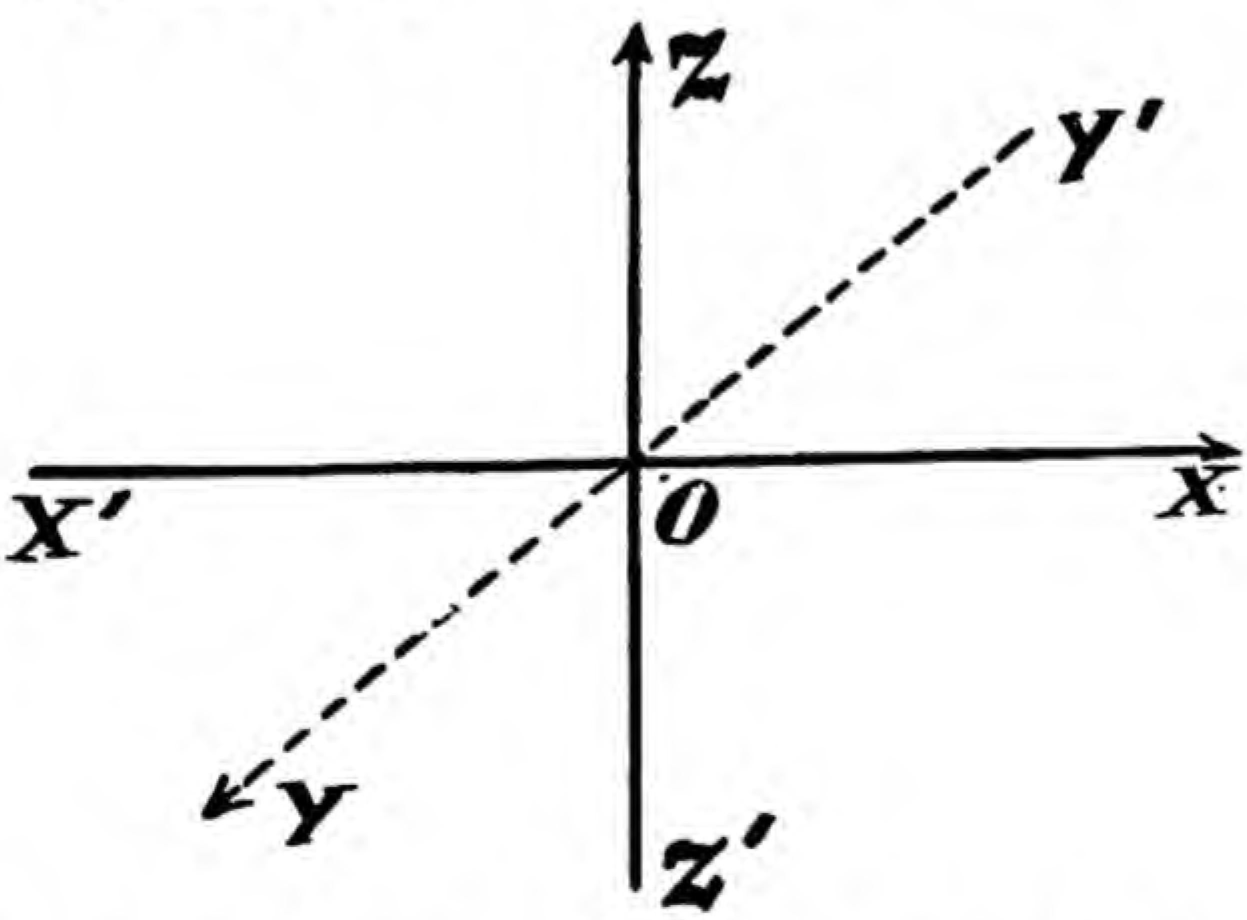
Contents

1

The Point in Space. Direction Cosines

1·1. Origin and Coordinate axes : Def.

Let X′OX and Z′OZ be two perpendicular straight lines meeting in O. Let Y′OY be a line through O perpendicular to the plane XOZ. The point O is called the **origin**, X′OX the axis of x, Y′OY the axis of y and Z′OZ the axis of z.

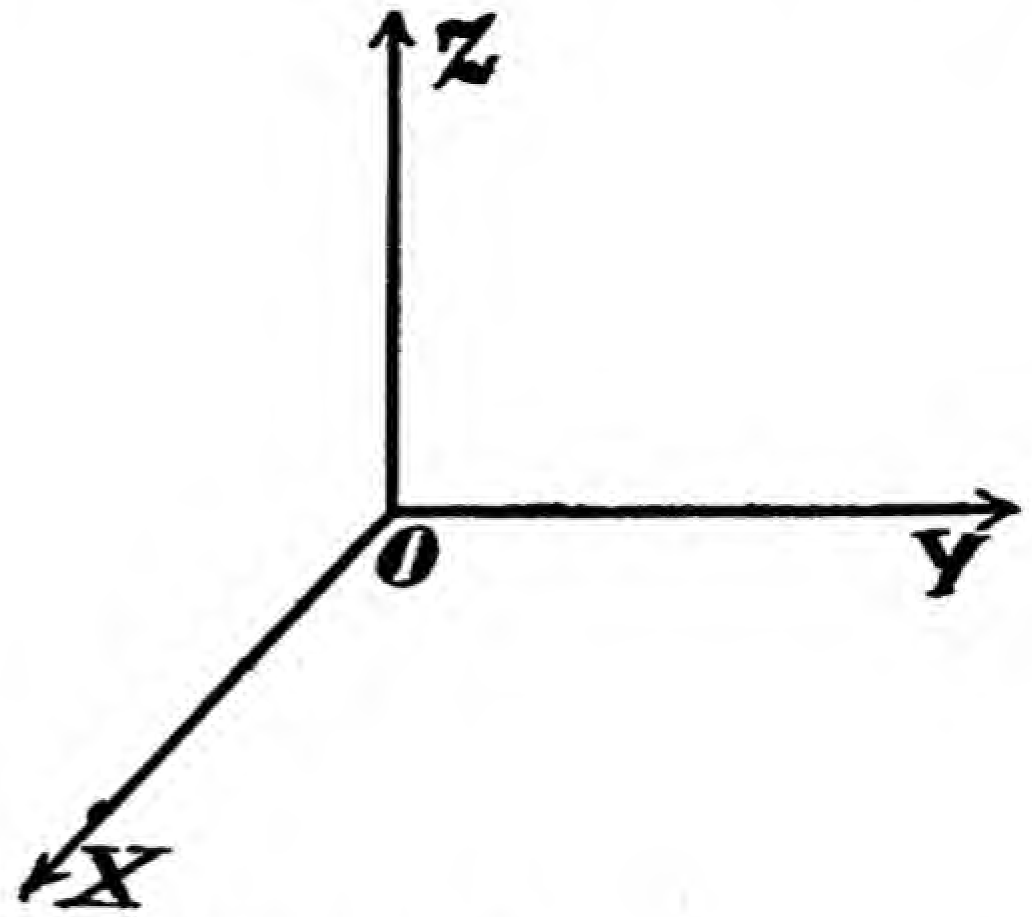


Note 1. *X′OX, Y′OY and Z′OZ are called the coordinate axes, or simply the axes.*

Note 2. *If the axes are mutually perpendicular, they are called rectangular axes, otherwise oblique axes. Unless otherwise is mentioned, we shal ltake the axes to be rectangular.*

Note. 3. *The positive directions of the axes are X′OX, Y′OY, Z′OZ and are indicated by the arrows.*

Note 4. *Some writers take the rectangular axes of x, y and z as shown in the following diagram. In this book, for classical treatment we shall adhere to the old practice and take the axes as shown in Art. 1·1, and for vector treatment we shall take the right-handed system of rectangular axes as shown in the following figure.*

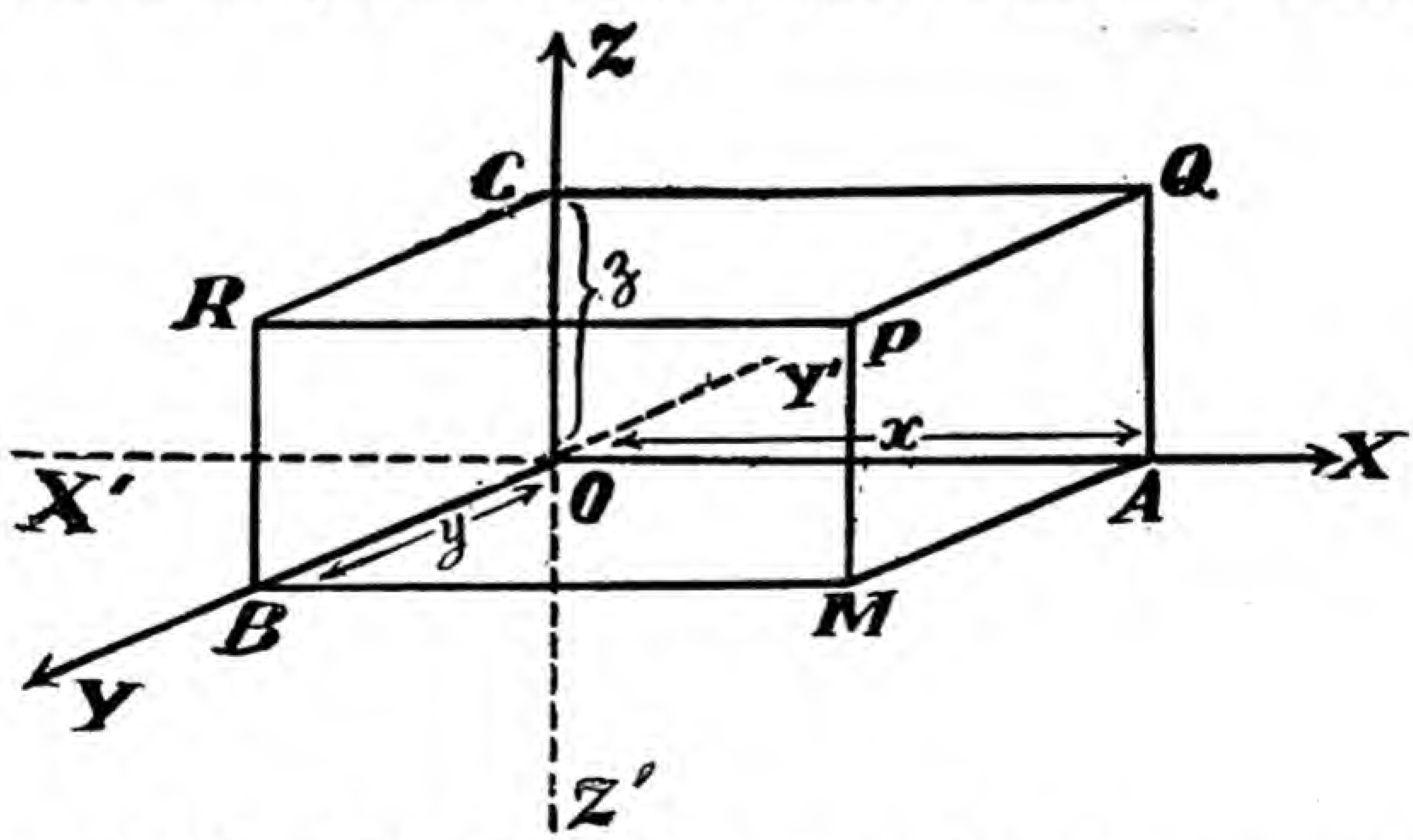


1·2. Coordinate planes : Def.

The plane YOZ (containing the axes of y and z) is called **yz-plane.** Similarly, the planes ZOX and XOY containing z, x axes and y, x axes respectively are called **zx-plane** and **xy-plane.** These planes together are called the **coordinate planes.**

1·3. Coordinates of a point in space : Def.

Let P be **any** point in space. Through P draw a plane PQAM parallel to the yz-plane cutting OX in A. Also, through P draw



planes PRBM and PRCQ parallel to zx- and xy-planes respectively cutting OY in B and OZ in C. If $OA=x$, $OB=y$, $OC=z$, then the three numbers x, y, z taken with proper signs are called the coordinates of P, and are written as (x, y, z).

Thus, *the coordinates of a point in space are the distances of the origin from the points of the axes in which the planes through that point parallel to the coordinate planes intersect the axes.*

Note 1. Another explanation of coordinates of a point in space. (*i*) *In the above figure the three coordinate planes and the three planes through P parallel to the coordinate planes form a rectangular parallelopiped whose 6 rectangular faces are in parallel pairs, as (BOCR, MAQP), (OAQC, BMPR) and (OBMA, CRPQ).*

$\therefore$ *x-coordinate of* $P = OA = CQ = RP$
= length of the perpendicular from P on yz-plane,

y-coordinate of $P = OB = AM = QP$
= length of the perpendicular from P on zx-plane,

and *z-coordinate of* $P = OC = AQ = MP$
= length of the perpendicular from P on xy-plane.

$\therefore$ **the coordinates of a point P in space are the distances of P from the three coordinate planes yz, zx and xy respectively.**

(*ii*) *The plane PMAQ is parallel to the yz-plane and is therefore perpendicular to the x-axis.*

Now PA lies in the plane PMAQ.

$\therefore$ $PA \perp OX$.

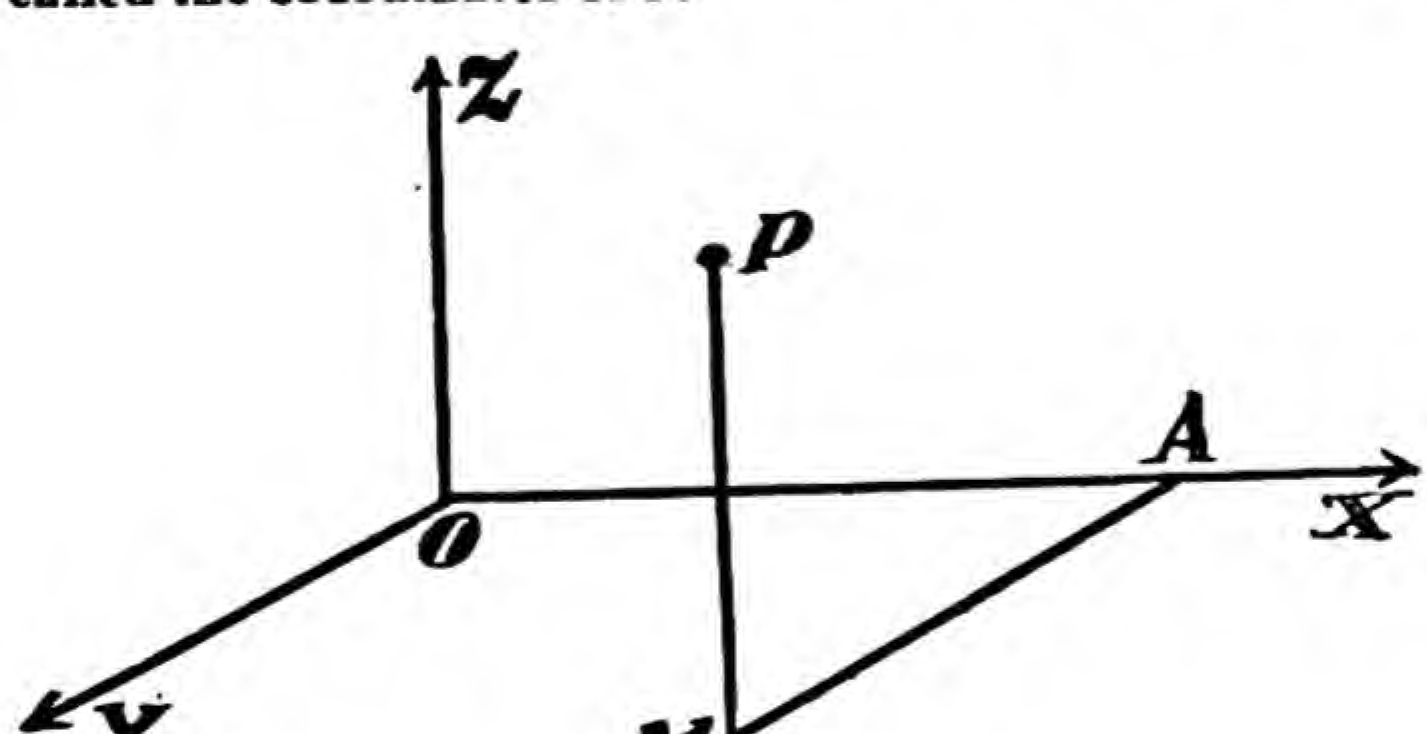
Similarly, $PB \perp OY$ *and* $PC \perp OZ$.

$\therefore$ *x-coordinate of* $P = OA$, *y-coordinate of* $P = OB$ *and*
z-coordinate of $P = OC$, *where A, B, C are the feet of the perpendiculars from P on the axes of x, y and z respectively.*

$\therefore$ **the coordinates of a point P in space are the distances from origin of the feet of the perpendiculars A, B and C from P on the axes respectively.**

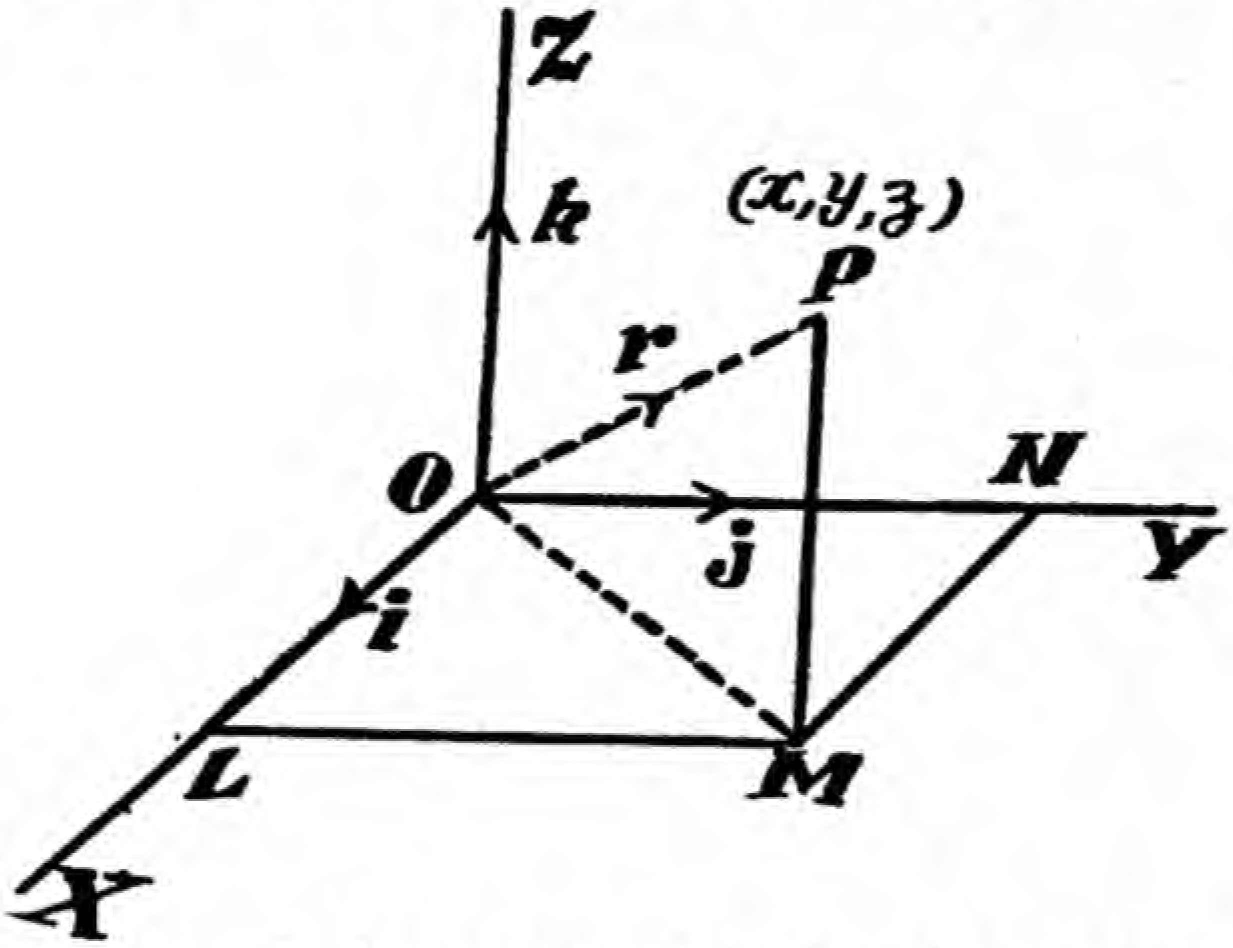
(*iii*) $MP = AQ = OC =$ *z-coordinate of P,*
$AM = OB =$ *y-coordinate of P,*
and $OA =$ *x-coordinate of P.*

$\therefore$ **if PM be drawn perpendicular to xy-plane meeting it at M and through M, MA be drawn parallel YO to meet OX in A, then (OA, AM, MP) are called the coordinates of P.**



Note 2. Position vector of a point whose cartesian coordinates are (x, y, z).

Let P be the point (x, y, z). *Let the vector* $\overrightarrow{OP}$ *be* $\mathbf{r}$.



Draw PM perpendicular to the xy-plane. Draw MN ond ML porallel to OX and YO respectively. Join OM.

Now, $\mathbf{r}=\overrightarrow{OP}=\overrightarrow{OM}+\overrightarrow{MP}=\overrightarrow{OL}+\overrightarrow{ON}+\overrightarrow{MP}$ ...(*I*)

Let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ *be the unit vectors in the directions OX, OY and OZ respectively.*

Now, $OL=x$, $LM=ON=y$ *and* $MP=z$.

$\therefore$ $\overrightarrow{OL}=x\mathbf{i}$, $\overrightarrow{ON}=y\mathbf{j}$ *and* $\overrightarrow{MP}=z\mathbf{k}$,

$\therefore$ (*I*) *becomes* $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$.

Thus if the coordinates of a point are (x, y, z), *its position vector is* $x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$.

Note 3. Origin. *The coordinates of the origin are* $(0, 0, 0)$.

Note 4. Convention of signs.

The x-coordinate, y-coordinate and the z-coordinate are considered positive if they are measured in the direction OX, OY and OZ respectively.

Note 5. Octants. *The three coordinate planes divide the whole of the space into 8 parts called* **octant.** *The octant OXYZ, in which all the three coordinates are positive, is called the first octant.*

SECTION I

DISTANCE BETWEEN TWO POINTS

1·4. Distance formula.

To find the distance between two points whose coordinates are $(\mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1)$ **and** $(\mathbf{x}_2, \mathbf{y}_2, \mathbf{z}_2)$, **the axes being rectangular.**

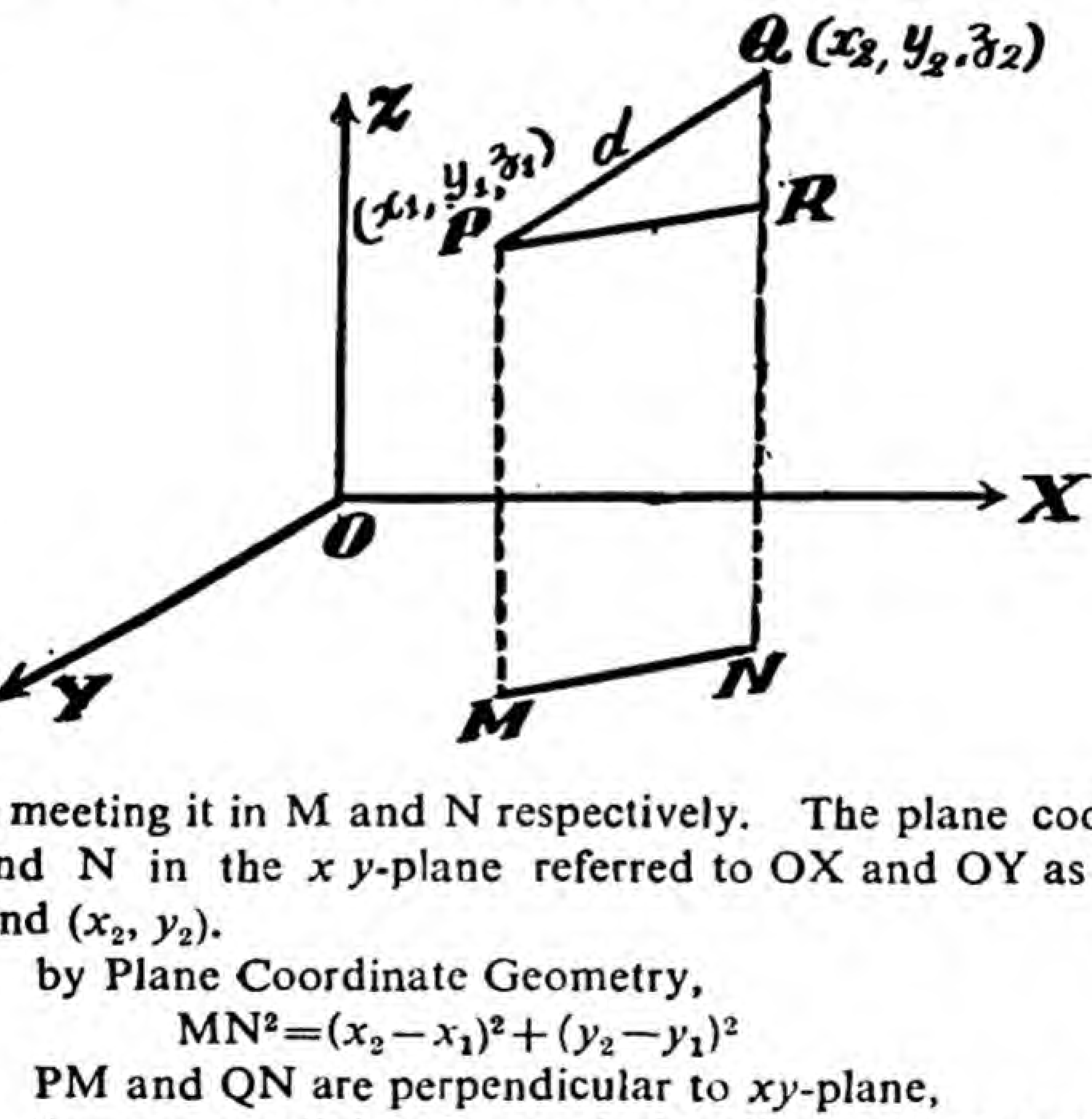
Let $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ be the position vectors of the given points P and Q whose rectangular cartesian coordinates are (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively.

$$\therefore \quad \overrightarrow{OP}=x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k} \text{ and } \overrightarrow{OQ}=x_2\mathbf{i}+y_2\mathbf{j}+z_2\mathbf{k}.$$

Now, $\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}=(x_2-x_1)\mathbf{i}+(y_2-y_1)\mathbf{j}+(z_2-z_1)\mathbf{k}$

$$\therefore \quad PQ=|\overrightarrow{PQ}|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}.$$

Aliter. Let P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) be the given points. Let PQ$=d$. Through P and Q draw PM and QN perpendiculars to



xy-plane meeting it in M and N respectively. The plane coordinates of M and N in the xy-plane referred to OX and OY as axes are (x_1, y_1) and (x_2, y_2).

$\therefore$ by Plane Coordinate Geometry,

$$MN^2=(x_2-x_1)^2+(y_2-y_1)^2 \qquad \ldots(1)$$

$\because$ PM and QN are perpendicular to xy-plane,

$\therefore$ they are parallel and as such they lie in the same plane.

$\therefore$ if we draw a line through P parallel to MN, then it also lies in this plane and therefore meet QN in the point R and is also perpendicular to it.

From the right-angled triangle PRQ, we have

$$PQ^2=PR^2+QR^2=MN^2+(QN-RN)^2$$
$$=MN^2+(QN-PM)^2$$

But $\quad PM = z_1 \quad$ and $QN = z_2$.

$\therefore \quad PQ^2 = (x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2$

or
$$\mathbf{d} = \sqrt{(\mathbf{x}_2-\mathbf{x}_1)^2 + (\mathbf{y}_2-\mathbf{y}_1)^2 + (\mathbf{z}_2-\mathbf{z}_1)^2},$$
which is the required distance.

Cor. Distance from the origin of a point $(\mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1)$.

The distance of the point (x_1, y_1, z_1) from the origin is
$$\sqrt{\mathbf{x}_1^2 + \mathbf{y}_1^2 + \mathbf{z}_1^2}.$$

Aid to memory. Distance between two points
$$= \sqrt{(\text{Diff. of x-Coor.}^2) + (\text{Diff. of y-Coor.})^2 + (\text{Diff. of z-Coor})^2}.$$

EXAMPLES I (A)

Ex. 1. If A and B be the points (3, 4, 5) and (−1, 3, −7) respectively, find the locus of a point P such that
$$\mathbf{PA^2 + PB^2 = k^2}.$$

Sol. Let P be (λ, μ, ν).

By the condition of the problem,
$$(\lambda-3)^2 + (\mu-4)^2 + (\nu-5)^2 + (\lambda+1)^2 + (\mu-3)^2 + (\nu+7)^2 = k^2,$$
or,
$$2\lambda^2 + 2\mu^2 + 2\nu^2 - 4\lambda - 14\mu + 4\nu + 109 = k^2.$$

$\therefore$ locus of (λ, μ, ν) is
$$2x^2 + 2y^2 + 2z^2 - 4x - 14y + 4z + 109 - k^2 = 0.$$

Ex. 2. Prove that the points (3, −2, 4), (1, 1, 1) and (−1, 4, 2) are collinear.

Ex. 3. Show that the points (3, 2, 2), (−1, 1, 3), (0, 5, 6), (2, 1, 2) lie on a sphere whose centre is (1, 3, 4).

Also find its radius. [**Ans.** 3.]

Ex. 4. Find the locus of points which are equidistant from the points, (1, 2, 3) and (3, 2, 1). *(Bihar, 1961 S)*

[**Ans.** $x - 2z = 0$.]

SECTION II

SECTION FORMULAE

1·5. To find the coordinates of the point which divides internally the straight line joining two given points $(\mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1)$ and $(\mathbf{x}_2, \mathbf{y}_2, \mathbf{z}_2)$ in the given ratio $\mathbf{m}_1 : \mathbf{m}_2$. *(Bihar, 1960)*

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be the given points. Let the point $R(x, y, z)$ divide the line joining P and Q in the given ratio $m_1 : m_2$.

Let **a, b** and **r** be the position vectors of the points P, Q and R respectively.

$\therefore$
$$\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k},$$
$$\mathbf{b} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k},$$
and
$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

where **i**, **j**, **k** are unit vectors in the directions of the positive axes of x, y and z.

Now, $$\frac{PR}{RQ}=\frac{m_1}{m_2}$$

$\therefore$ $$m_2 \,.\, PR = m_1 \,.\, RQ,$$

or, $$m_2 \,.\, \vec{PR} = m_1 \,.\, \vec{RQ},$$

or, $$m_2\,(\vec{PO}+\vec{OR}) = m_1\,(\vec{RO}+\vec{OQ}),$$

or, $$m_2\,(\vec{OR}-\mathbf{a}) = m_1(-\vec{OR}+\mathbf{b}),$$

or, $$(m_1+m_2)\,\vec{OR} = m_1\mathbf{b}+m_2\mathbf{a},$$

or, $$x\mathbf{i}+y\mathbf{j}+z\mathbf{k}=\frac{[m_1(x_2\mathbf{i}+y_2\mathbf{j}+z_2\mathbf{k})+m_2(x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k}]}{m_1+m_2}$$

$$=\frac{[(m_1x_2+m_2x_1)\mathbf{i}+(m_1y_2+m_2y_1)\mathbf{j}+(m_1z_2+m_2z_1)\mathbf{k}]}{m_1+m_2},$$

or, $$x\mathbf{i}+y\mathbf{j}+z\mathbf{k}=\frac{m_1x_2+m_2x_1}{m_1+m_2}\,\mathbf{i}+\frac{m_1y_2+m_2y_1}{m_1+m_2}\,\mathbf{j}+\frac{m_1z_2+m_2z_1}{m_1+m_2}\,\mathbf{k}$$

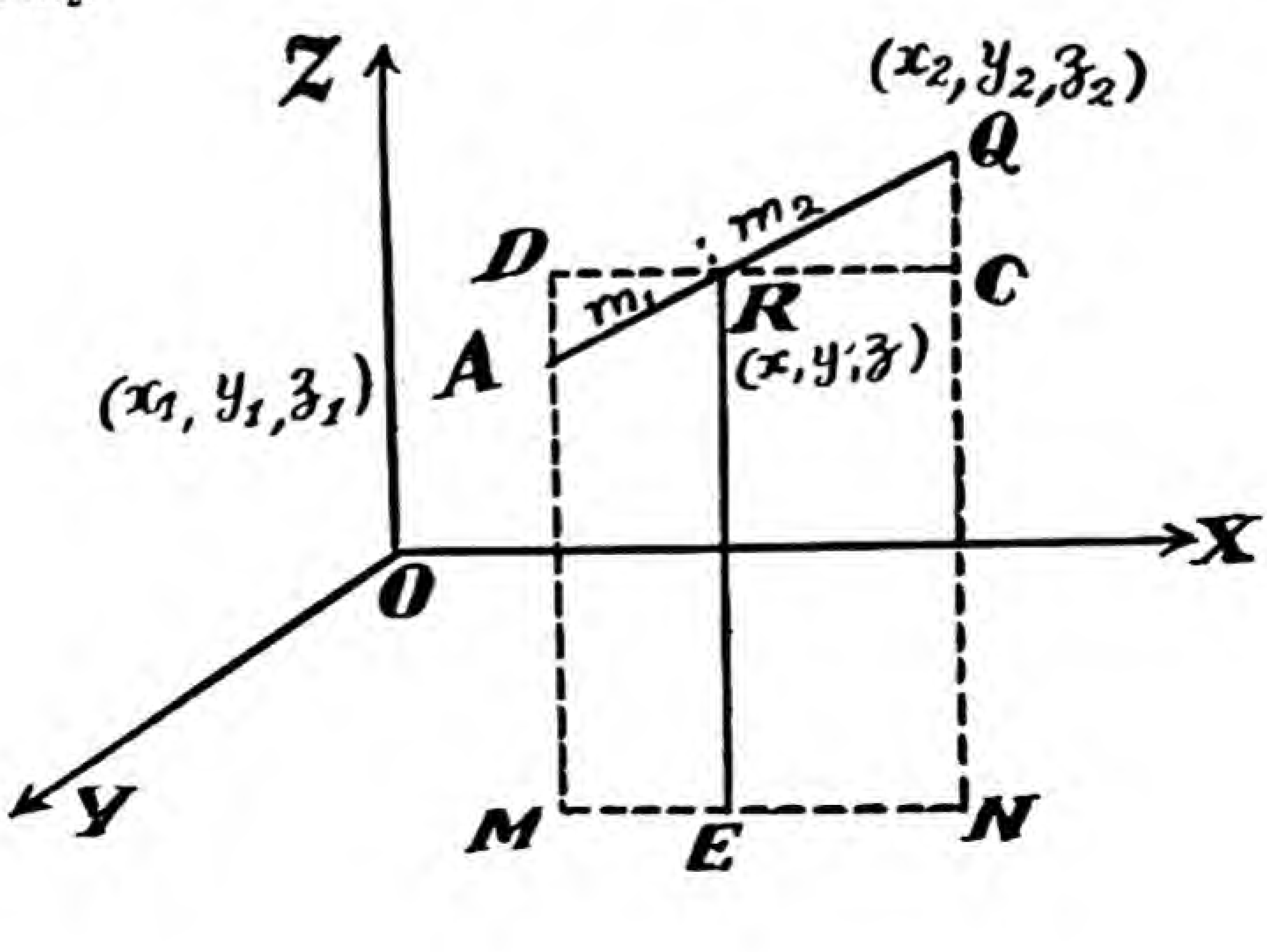
Equating coefficients of like vectors, we have

$$x=\frac{m_1x_2+m_2x_1}{m_1+m_2},\ y=\frac{m_1y_2+m_2y_1}{m_1+m_2},\ z=\frac{m_1z_2+m_2z_1}{m_1+m_2}.$$

Hence the coordinates of R are

$$\left[\frac{m_1x_2+m_2x_1}{m_1+m_2},\ \frac{m_1y_2+m_2y_1}{m_1+m_2},\ \frac{m_1z_2+m_2z_1}{m_1+m_2}\right].$$

Aliter. Let A(x_1, y_1, z_1) and Q (x_2, y_2, z_2) be the given points. Let the point R (x, y, z) divide the line joining A and Q in the given ratio $m_1 : m_2$.



Through A, Q, R draw perpendiculars AM, QN and RE to the xy-plane.

$\because$ these lines are all perpendicular to the xy-plane,

$\therefore$ they are parallel.

$\because$ they are cut by the same line **ARQ**,

$\therefore$ they lie in the same plane.

$\therefore$ if we draw through R a line parallel to MEN it will lie in the same plane and as such it cuts MA produced in D and NQ in C.

Now, from the similar triangles ADR and RCQ,

$$\frac{m_1}{m_2}=\frac{AR}{RQ}=\frac{AD}{CQ}=\frac{DM-AM}{QN-CN}=\frac{RE-AM}{QN-RE}=\frac{z-z_1}{z_2-z},$$

or, $$m_1z_2-m_1z=m_2z-m_2z_1,$$

or, $$z(m_1+m_2)=m_1z_2+m_2z_1,$$

$\therefore$ $$z=\frac{\mathbf{m_1z_2+m_2z_1}}{\mathbf{m_1+m_2}}.$$

Similarly, by drawing perpendiculars to yz- and zx- planes and proceeding as above, we can show that

$$\mathbf{x}=\frac{\mathbf{m_1x_2+m_2x_1}}{\mathbf{m_1+m_2}}$$

and $$\mathbf{y}=\frac{\mathbf{m_1y_2+m_2y_1}}{\mathbf{m_1+m_2}}.$$

Hence **the coordinates of R are**

$$\left[\frac{\mathbf{m_1x_2+m_2x_1}}{\mathbf{m_1+m_2}},\ \frac{\mathbf{m_1y_2+m_2y_1}}{\mathbf{m_1+m_2}},\ \frac{\mathbf{m_1z_2+m_2z_1}}{\mathbf{m_1+m_2}}\right].$$

Note 1. $\frac{m_1}{m_2}$ *is positive, then R divides AQ internally, and if* $\frac{m_1}{m_2}$ *is negative, R divides AQ externally.*

Note 2. Middle point formula.

To find the coordinates of the middle point of the straight line joining the points ($\mathbf{x_1, y_1, z_1}$) and ($\mathbf{x_2, y_2, z_2}$).

If $m_1=m_2$, *R becomes the middle point of AQ.*

$\therefore$ *its coordinates are* $\left[\frac{m_1x_2+m_1x_1}{m_1+m_1},\ \frac{m_1y_2+m_1y_1}{m_1+m_1},\ \frac{m_1z_2+m_1z_1}{m_1+m_1}\right]$,

or, $$\left[\frac{\mathbf{x_1+x_2}}{2},\ \frac{\mathbf{y_1+y_2}}{2},\ \frac{\mathbf{z_1+z_2}}{2}\right].$$

EXAMPLES I (B)

Ex. 1. Given that P (3, 2, −4), Q(5, 4, −6), R(9, 8, −10) are collinear, find the ratio in which Q divides PR.

[Punjab (Pakistan), B. Sc., 1956 S.]

Sol. Let Q divide PR in the ratio $\lambda:1$.

$\therefore$ $5=\frac{3.1+9.\lambda}{\lambda+1}$ or, $5\lambda+5=3+9\lambda$,

or, $4\lambda=2$ $\therefore$ $\lambda=\frac{1}{2}$.

$\therefore$ the required ratio is 1 : 2.

Ex. 2. Find the centroid of the triangle whose vertices are (x_r, y_r, z_r), $r=1, 2, 3$ is $[\frac{1}{3}(x_1+x_2+x_3), \frac{1}{3}(y_1+y_2+y_3), \frac{1}{3}(z_1+z_2+z_3)]$.

Ex. 3. Centre of gravity of tetrahedron.

Show that the centroid of the tetrahedron whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) is $[\frac{1}{4}(x_1+x_2+x_3+x_4), \frac{1}{4}(y_1+y_2+y_3+y_4), \frac{1}{4}(z_1+z_2+z_3+z_4)]$.

Ex. 4. From the point $(1, -2, 3)$ lines are drawn to meet the sphere $x^2+y^2+z^2=4$, and they are divided in the ratio 2 : 3. Prove that the points of section lie on the sphere

$$5x^2+5y^2+5z^2-6x+12y-18z+22=0$$

(Delhi Hons., 1963)

Sol. Let the line through $(1, -2, 3)$ meet the sphere in the point (λ, μ, ν^2).

$$\therefore \quad \lambda^2+\mu^2+\nu^2=4 \quad \ldots(1)$$

Let (α, β, γ) be the point which devides the join of $(1, -2, 3)$ and (λ, μ, ν) in the ratio 2 : 3.

$$\therefore \quad \alpha=\frac{2\lambda+3}{2+3}, \quad \text{or,} \quad \lambda=\frac{5\alpha-3}{2},$$

$$\beta=\frac{2\mu-6}{2+3}, \quad \text{or,} \quad \mu=\frac{5\beta+6}{2}.$$

and

$$\gamma=\frac{2\nu+9}{2+3}, \quad \text{or,} \quad \nu=\frac{5\gamma-9}{2}.$$

Substituting these values of λ, μ, ν in (1) we have

$$(5\alpha-3)^2+(5\beta+6)^2+(5\gamma-9)^2=16,$$

or,

$$25\alpha^2+25\beta^2+25\gamma^2-30\alpha+60\beta-90\gamma+110=0,$$

or,

$$5\alpha^2+5\beta^2+5\gamma^2-6\alpha+12\beta-18\gamma+22=0.$$

$\therefore$ locus of (α, β, γ) is

$$5x^2+5y^2+5z^2-6x+12y-18z+22=0.$$

Ex. 4. (i) Find the point where the line joining $(2, -3, 1)$ and $(3, -4, -5)$ cuts the plane $2x+y+z=7$.

(Bihar, 1961 S ; Bhagalpur, 1962).

[**Ans.** $(1, -2, 7)$.]

(*ii*) Find the ratio in which the join of $(2, 1, 5)$, $(3, 4, 3)$ is divided by the plane $x+y-z=\frac{1}{2}$. *(Bhagalpur, 1963)*

[**Ans.** 5 : 7]

SECTION III

DIRECTION COSINES AND PROJECTION OF A LINE

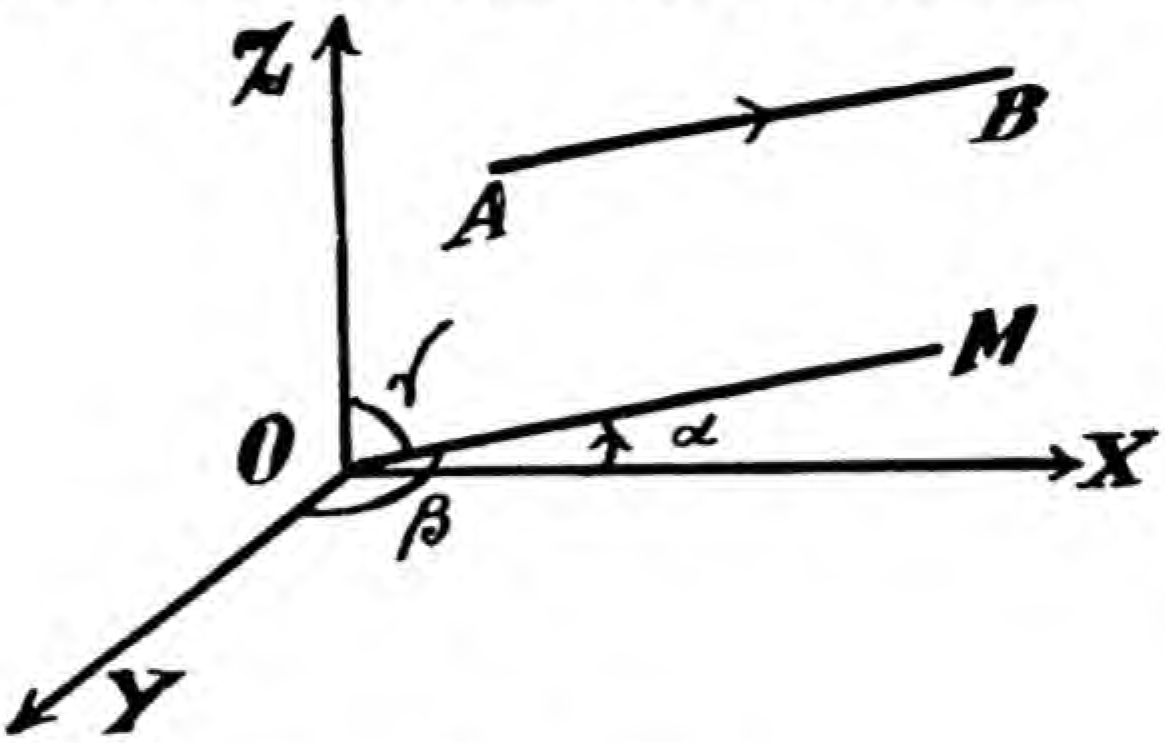
1·6. Angle between two non-intersecting lines : Def.

The angle between two non-intersecting lines is the angle between two straight lines drawn through any point in space parallel to them.

1·7. Direction cosines of a straight line : Def.

If α, β, γ be the angles which a straight line AB makes with the **positive directions of axes** of x, y, z respectively, then $\cos\alpha$, $\cos\beta$,

$\cos\gamma$ are called the **direction cosines** of the line AB, and are usually denoted by the letters l, m and n respectively.



Note 1. $l=\cos\alpha$, $m=\cos\beta$, $n=\cos\gamma$.

Note 2. Direction cosines of BA.

$\because$ *BA makes* $180^\circ-\alpha$, $180^\circ-\beta$, $180^\circ-\gamma$, $\therefore$ *the direction cosines of BA are* $\cos(180^\circ-\alpha)$, $\cos(180^\circ-\beta)$ *and* $\cos(180^\circ-\gamma)$ *or* $-\cos\alpha, -\cos\beta, -\cos\gamma$, *or*, $-l, -m, -n$.

Note 3. Direction cosines of the axes.

The x-axis makes angles of 0°, 90°, 90° *with the axes of x, y and z respectively.*

$\therefore$ *the* **direction cosines of the x-axis are 1, 0, 0.** *Similarly, the* **direction cosines of the y-axis and z-axis are respectively 0, 1, 0 and 0, 0, 1.**

Note 4. *Let OM be r.*

$$\therefore\ \mathbf{r}=r\hat{\mathbf{r}},\quad or,\quad \hat{\mathbf{r}}=\frac{\mathbf{r}}{r}=\frac{x}{r}\,\mathbf{i}+\frac{y}{r}\,\mathbf{j}+\frac{z}{r}\,\mathbf{k},$$

where (x, y, z) *are the coordinates of M.*

$$=(\cos\alpha)\,\mathbf{i}+(\cos\beta)\,\mathbf{j}+(\cos\gamma)\mathbf{k}.$$

Hence '**if a unit vector is resolved in terms of i, j, and k, then the coefficients of i, j, k are the direction cosines of that vector.**

1·8. If l, m, n are the direction cosines of a line OP, where OP = r, to show that the coordinates of P are (lr, mr, nr), O being the origin.

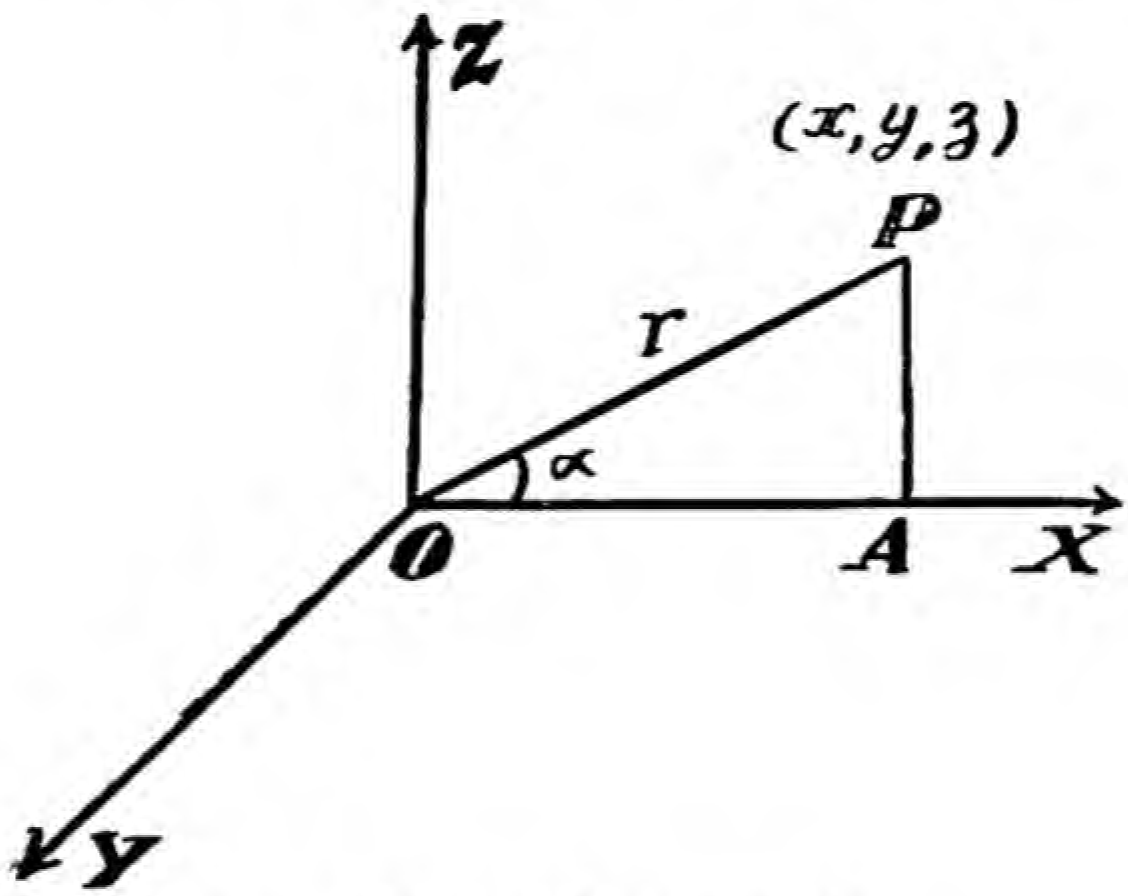
Proof. Let P be the point (x, y, z). Draw PA perpendicular to the x-axis meeting it in A.

$$\therefore\qquad OA=x.$$

From the right-angled triangle OAP, we have $OA=OP\cos\alpha$, where α is the angle which OP makes with OX,

or, $$x=r\cos\alpha=rl=lr.$$

Similarly, $y=mr$, $z=nr$.



$\therefore$ the coordinates of P are (lr, mr, nr).

This proves the proposition.

1·9. An important relation. If l, m, n are the direction cosines of a straight line, to prove that $l^2+m^2+n^2=1$.

Proof.

Let $\overrightarrow{OP}$ be $\mathbf{r}$ and $|\overrightarrow{OP}|$ be r.

Let the coordinates of P be (x, y, z).

$\therefore \quad \mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}=lr\mathbf{i}+mr\mathbf{j}+nr\mathbf{k} \quad (\because\ x=lr,\ y=mr,\ z=nr.)$

$\therefore \quad r^2=\mathbf{r}^2=(lr\mathbf{i}+mr\mathbf{j}+nr\mathbf{k})^2,$

or, $\quad r^2=r^2(l^2+m^2+n^2), \quad (\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=1,\ \mathbf{i}.\mathbf{j}=\mathbf{j}.\mathbf{k}=\mathbf{k}.\mathbf{i}=0).$

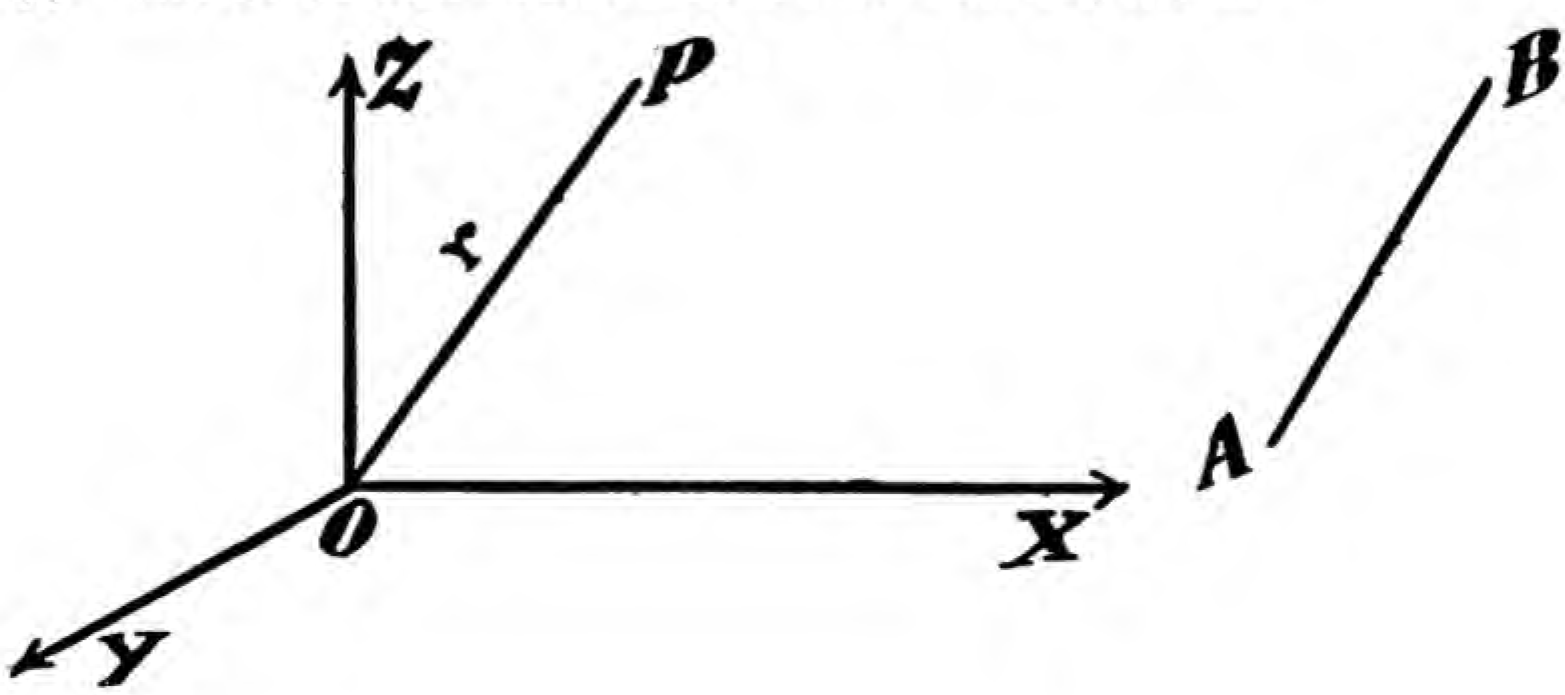
or, $\quad l^2+m^2+n^2=1.$ This proves the proposition.

Aliter.

Let l, m, n be the direction-cosines of the line AB.

Through O draw a straight line OP parallel to AB.

$\therefore$ the direction cosines of OP are also l, m, n.



Let OP be r. Let the coordinates of P be (x, y, z).

$\therefore \quad x=lr,\ y=mr,\ z=nr.$ (Art. 1·8).

$\therefore \quad x^2+y^2+z^2=r^2\,(l^2+m^2+n^2),$

or, $\quad r^2=r^2\,(l^2+m^2+n^2) \qquad (\because \ r^2=\text{OP}^2=x^2+y^2+z^2)$

$\therefore \quad \mathbf{l^2+m^2+n^2=1}.$ This proves the proposition.

1·10. Direction ratios : Def.

A set of three numbers a, b, c which are proportional to the direction cosines l, m, n respectively of a line are called the **direction ratios** of that line.

Note. How to find the direction cosines of a line, if its direction ratios are given ?

Let l, m, n be the actual direction cosines and a, b, c be the direction ratios of a line

$$\therefore \quad \frac{l}{a}=\frac{m}{b}=\frac{n}{c}=\frac{\pm\sqrt{l^2+m^2+n^2}}{\sqrt{a^2+b^2+c^2}}=\frac{\pm 1}{\sqrt{a^2+b^2+c^2}}.$$

$$\therefore \quad l=\pm\frac{a}{\sqrt{a^2+b^2+c^2}},\ m=\pm\frac{b}{\sqrt{a^2+b^2+c^2}}$$

$$n=\pm\frac{c}{\sqrt{a^2+b^2+c^2}}.$$

If the senses of the line are ignored, the actual direction cosines of the line are

$$\frac{a}{\sqrt{a^2+b^2+c^2}},\quad \frac{b}{\sqrt{a^2+b^2+c^2}},\quad \frac{c}{\sqrt{a^2+b^2+c^2}}.$$

Aid to memory. To obtain the direction cosines of a line from its direction ratios a, b, c, divide each of them by

$$\sqrt{\mathbf{a^2+b^2+c^2}}.$$

Caution. $\mathbf{a^2+b^2+c^2\neq 1}.$

1·11. Projection of a point on a line : Def.

The projection of a point on a line is the foot of the perpendicular from that point on the given line.

Note. *The projection of a point on a line is the point in which the plane, through that point perpendicular to the line meets it.*

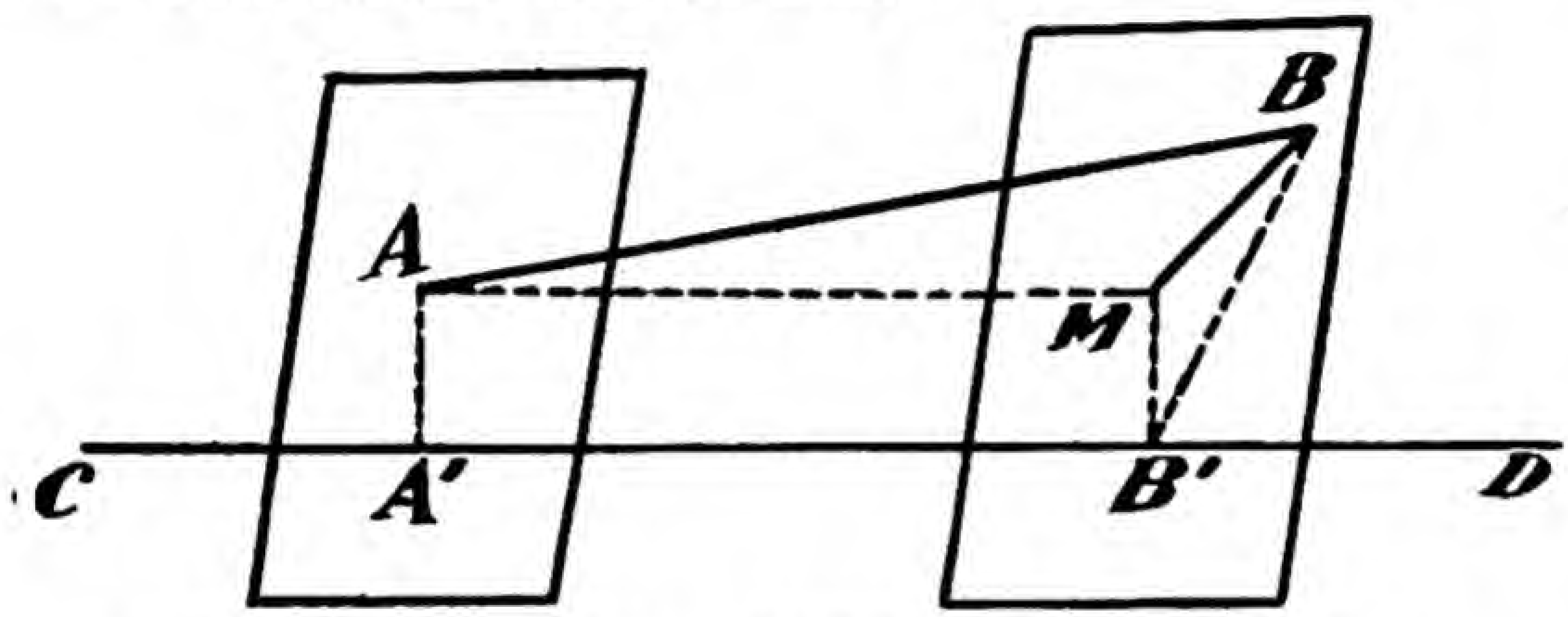
1·12. Projection of a segment of a line on another line : Def.

The projection of a segment of a line AB on another line CD is the segment PQ, where P and Q are the feet of the perpendiculars from A and B on CD.

1·13. Length of the projection of the segment of a line on another line.

To show that the projection of a segment AB of any line on another line CD is A′B′ such that A′B′ = AB cos θ, where θ is the angle between AB and CD.

Proof. Let A′ and B′ be the points where the planes through A and B meets the line CD respectively.



Now, A′B′ = projection of AB on CD.

Through A draw a straight line parallel to CD to meet the plane through B perpendicular to CD at M.

$\because$ AM is $\parallel$ CD, $\therefore$ $\angle$MAB$=\theta$.

$\because$ BM lies in the plane which is perpendicular to AM,

$\therefore$ $\angle$AMB$=90°$.

From the right-angled triangle AMB, AM$=$AB cos θ.

But AM$=$A′B′.

Hence **A′B′$=$AB cos θ.** This prove the proposition.

Note. *If a broken continuous line consists of a number of segments P_1P_2, P_2P_3,......,P_{n-1} P_n, then the sum of the projections of all the segments, P_1P_2, P_2P_3,......, P_{n-1} P_n an any line CD is the same as the projection of the line P_1P_n on CD.*

This is left as an exercise for the students to verify.

1·14. Direction cosines of the line joining two points. To find the direction cosines of the line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2).

Let AB be the straight line joining the points A (x_1, y_1, z_1) and B (x_2, y_2, z_1) respectively.

Let O be the origin of reference.

$$\text{Now,}\quad \hat{\mathbf{AB}}=\frac{\overrightarrow{\mathbf{AB}}}{\text{AB}}=\frac{\overrightarrow{\mathbf{OB}}-\overrightarrow{\mathbf{OA}}}{\text{AB}}=\frac{(x_2-x_1)\,\mathbf{i}+(y_2-y_1)\mathbf{j}+(z_2-z_1)\mathbf{k}}{\text{AB}}$$

$$=\frac{(x_2-x_1)}{\text{AB}}\mathbf{i}+\frac{(y_2-y_1)}{\text{AB}}\mathbf{j}+\frac{(z_2-z_1)}{\text{AB}}\mathbf{k}$$

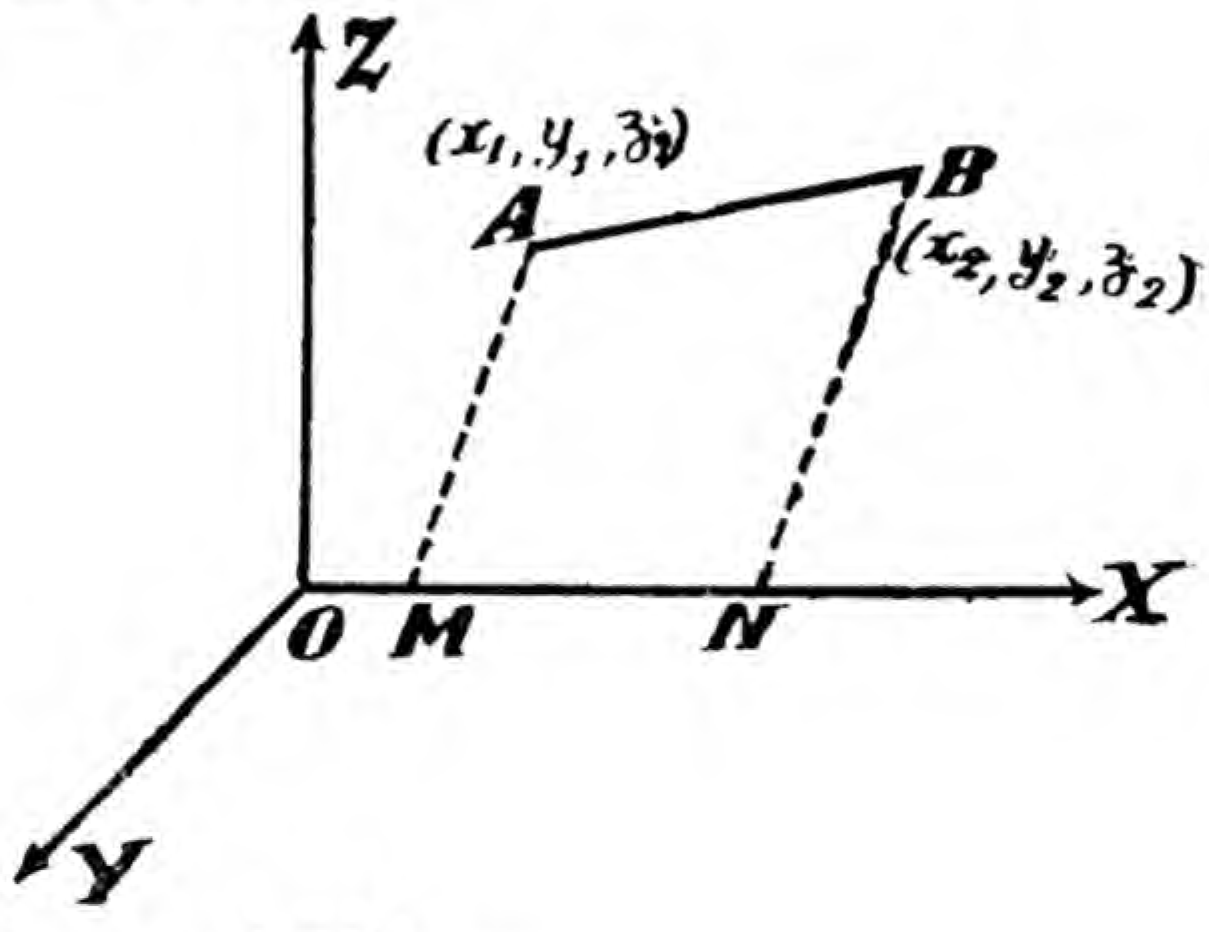
$$\therefore\qquad \left(\frac{x_2-y_1}{\text{AB}}\right),\ \left(\frac{y_2-y_1}{\text{AB}}\right)\ \text{and}\ \left(\frac{z_2-z_1}{\text{AB}}\right)$$

are the direction cosines of AB, using note 4, Art. 1·7.

Aliter.

Let l, m, n be the direction cosines of the line AB joining the points A (x_1, y_1, z_1) and B (x_2, y_2, z_2).

Let M and N be the feet of the perpendiculars from A and B on the x-axis respectively.



$\therefore\quad ON=x_2$ and $OM=x_1$.

Now $MN=ON-OM=x_2-x_1$.

Also, MN = projection of AB on OX

$=$AB cos α, where α is the angle which the line makes with OX. (Art. 1·13)

$=l.$AB.

$$\therefore\quad l.\text{AB}=x_2-x_1,\quad \text{or,}\quad l=\frac{x_2-x_1}{\text{AB}}.$$

$$\text{Similarly,}\quad m=\frac{y_2-y_1}{\text{AB}}\quad\text{and}\quad n=\frac{z_2-z_1}{\text{AB}}.$$

Hence **l, m, n are proportional to $\mathbf{x_2-x_1, y_2-y_1, z_2-z_1}$.**

Aid to memory. The direction cosines of a line joining two points are proportional to the differences of their x, y and z coordinates.

1·15. Projection of the line joining two points on a line.

To find the projection of the line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) on another line whose direction cosines are l, m, n.

Let OM be a line whose direction cosines are l, m, n.

Let OM be of unit length. Let A and B be the points (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively.

$\therefore$ $\overrightarrow{OM}=l\mathbf{i}+m\mathbf{j}+n\mathbf{k}$, where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the unit vectors along the axes of x, y and z of a right-handed system.

$$\text{Now,}\quad \overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=(x_2\mathbf{i}+y_2\mathbf{j}+z_2\mathbf{k})-(x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k})$$
$$=(x_2-x_1)\mathbf{i}+(y_2-y_1)\mathbf{j}+(z_2-z_1)\mathbf{k}$$

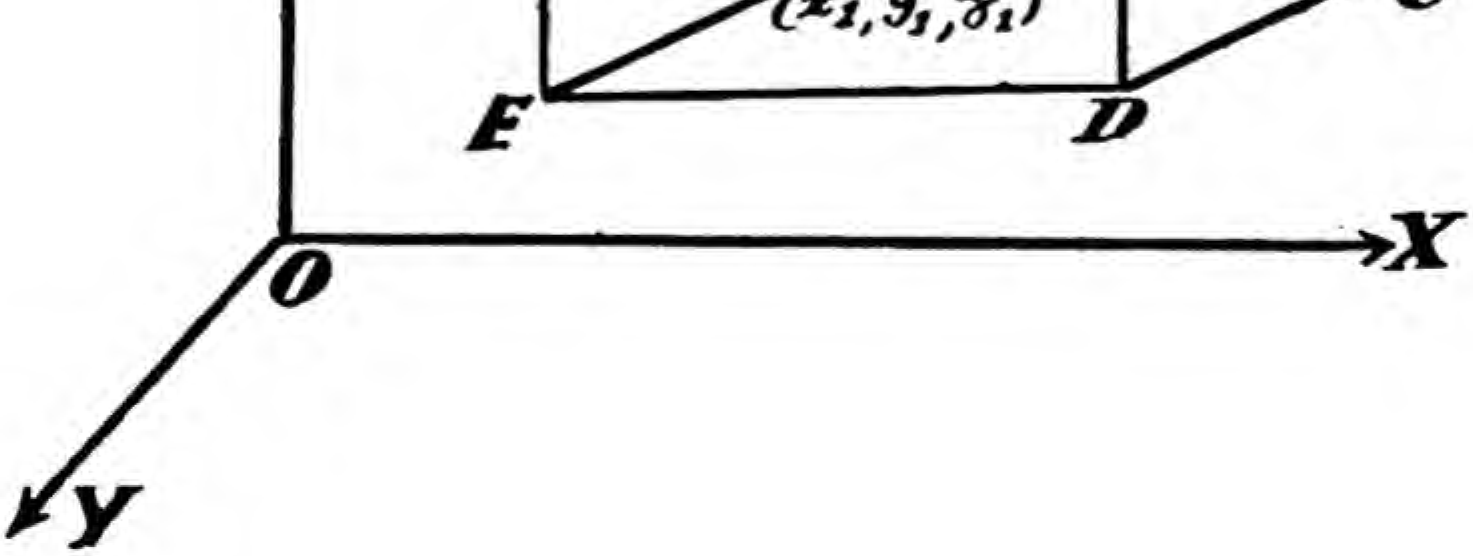
$$\text{The required projection}=\overrightarrow{AB}\cdot\overrightarrow{OM}$$
$$=[(x_2-x_1)\mathbf{i}+(y_2-y_1)\mathbf{j}+(z_2-z_1)\mathbf{k}]$$
$$*(l\mathbf{i}+m\mathbf{j}+n\mathbf{k})$$
$$=l(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1),$$
$$(\because\quad \mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=1,$$
$$\mathbf{i}.\mathbf{j}=\mathbf{j}.\mathbf{k}=\mathbf{k}.\mathbf{i}=0).$$

Aliter.

Let the given points be $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ and the given line be one having direction cosines l, m, n.

Through A and B draw planes parallel to the coordinates planes to form a rectangular parallelopiped with AB as its diagonal.

Now, $AC=x_2-x_1$, $CD=AE=y_2-y_1$, $DB=AG=z_2-z_1$.



The lines AC, CD and DB are parallel to the axes respectively.

$\therefore$ their projections on the line having direction cosines l, m, n are $(x_2-x_1)l$, $(y_2-y_1)m$, $(z_2-z_1)n$.

The lines AC, CD, and DB form a continuous broken line joining A and B.

$\because$ the sum of the projections of AC, CD and DB on any line is equal to the projection of AB on that line,

$\therefore$ the projection of AB on the line whose direction cosines are $l, m, n = (\mathbf{x_2}-\mathbf{x_1})\mathbf{l}+(\mathbf{y_2}-\mathbf{y_1})\mathbf{m}+(\mathbf{z_2}-\mathbf{z_1})\mathbf{n}$.

1·16. Angle formula.

To find the angle between two straight lines whose direction cosines are $\mathbf{l_1}$, $\mathbf{m_1}$, $\mathbf{n_1}$ and $\mathbf{l_2}$, $\mathbf{m_2}$, $\mathbf{n_2}$.

Let $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ be the unit vectors along the positive axes of x, y, and z respectively of a right-handed system. Let θ be the required angle between the lines AB and CD whose direction cosines are l_1, m_1, n_1 and l_2, m_2, n_2 respectively.

Through O draw lines OP, OQ parallel in the same sense to AB and CD respectively.

Let OP be r_1 and OQ be r_2.

Now, $\angle POQ=\theta$.

The coordinates of P and Q are $(l_1r_1,\ m_1r_1,\ n_1r_1)$ and (l_2r_2, m_2r_2, n_2r_2) respectively.

$$\therefore \quad \overrightarrow{OP}=l_1r_1\mathbf{i}+m_1r_1\mathbf{j}+n_1r_1\mathbf{k} \quad \text{and}$$

$$\overrightarrow{OQ}=l_2r_2\mathbf{i}+m_2r_2\mathbf{j}+n_2r_2\mathbf{k}.$$

$$\text{Now} \quad \overrightarrow{OP}.\overrightarrow{OQ}=(l_1r_1\mathbf{i}+m_1r_1\mathbf{j}+n_1r_1\mathbf{k}).(l_2r_2\mathbf{i}+m_2r_2\mathbf{j}+n_2r_2\mathbf{k}),$$

or $\quad r_1r_2\cos\theta=r_1r_2(l_1l_2+m_1m_2+n_1n_2), \quad \because \ \mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=1,$
$\mathbf{i}.\mathbf{j}=\mathbf{j}.\mathbf{k}=\mathbf{k}.\mathbf{i}=0.$

$$\therefore \quad \cos\theta=l_1l_2+m_1m_2+n_1n_2.$$

Aliter.

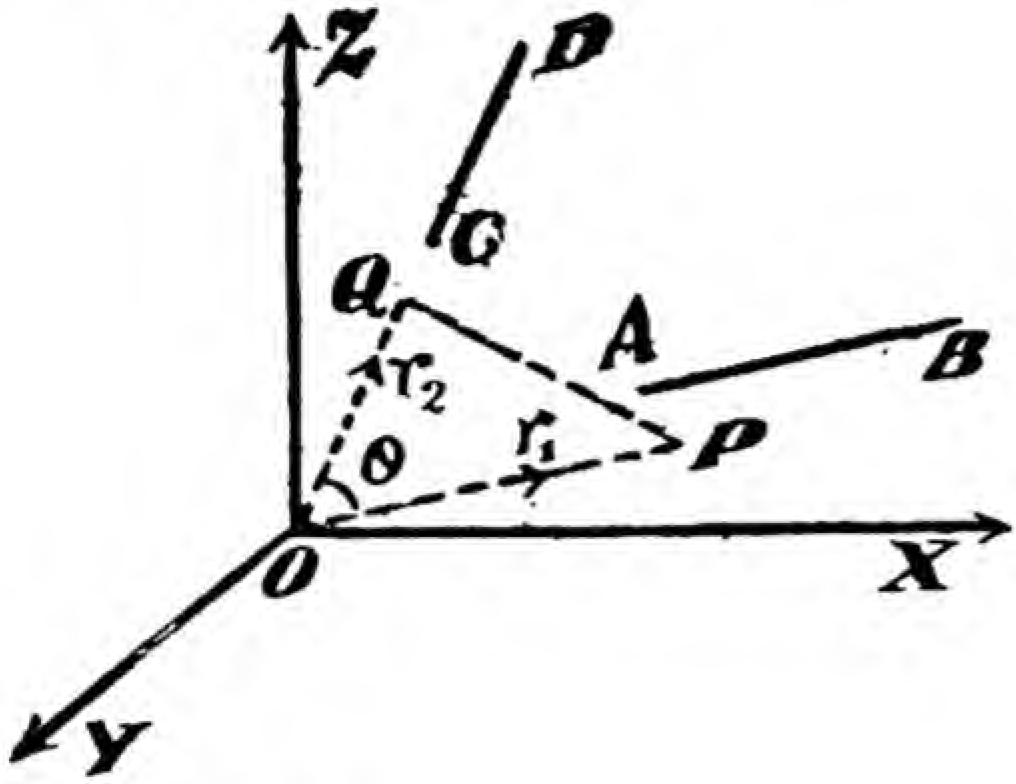
Let O be the angle between the lines AB and CD whose direction cosines are l_1, m_1, n_1 and l_2, m_2, n_2 respectively.

Through θ draw lines OP and OQ parallel in the same sense respectively to AB and CD.

Let OP be r_1 and OQ be r_2.

Now $\angle POQ=\theta$ (Art. 1·6).

The coordinates of P are (l_1r_1, m_1r_1, n_1r_1) and that of Q are (l_2r_2, m_2r_2, n_2r_2). Join PQ.



From the triangle OPQ,

$$PQ^2=OP^2+OQ^2-2OP.OQ\cos\theta,$$

or $$(l_1r_1-l_2r_2)^2+(m_1r_1-m_2r_2)^2+(n_1r_1-n_2r_2)^2 =r_1^2+r_2^2-2r_1r_2\cos\theta,$$

or, $$r_1^2(l_1^2+m_1^2+n_1^2)+r_2^2(l_2^2+m_2^2+n_2^2)-2r_1r_2(l_1l_2+m_1m_2+n_1n_2) =r_1^2+r_2^2-2r_1r_2\cos\theta,$$

or, $$r_1^2+r_2^2-2r_1r_2(l_1l_2+m_1m_2+n_1n_2)=r_1^2+r_2^2-2r_1r_2\cos\theta.$$

Hence $\mathbf{\cos\theta=l_1l_2+m_1m_2+n_1n_2}$.

Note. *Here l_1, m_1, n_1 and l_2, m_2, n_2 are* **actual direction cosines.**

1·17. Lagrange's identity.

To prove that $\mathbf{(l_1^2+m_1^2+n_1^2)(l_2^2+m_2^2+n_2^2)-(l_1l_2+m_1m_2+n_1n_2)^2=(m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2+(l_1m_2-l_2m_1)^2}$.

Proof.
$$\begin{aligned}\text{L.H.S.}&=l_1^2l_2^2+l_1^2m_2^2+l_1^2n_2^2+m_1^2l_2^2+m_1^2m_2^2+m_1^2n_2^2\\&\quad+n_1^2l_2^2+n_1^2m_2^2+n_1^2n_2^2-l_1^2l_2^2-m_1^2m_2^2-n_1^2n_2^3\\&\quad-2l_1m_1l_2m_2-2m_1m_2n_1n_2-2l_1l_2n_1n_2\\&=(m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2+(l_1m_2-l_2m_1)^2\\&=\text{R.H.S.}\end{aligned}$$

This proves the proposition.

1·18. To find (i) the sine of the angle, (ii) the tangent of the angle between the lines whose direction cosines are $\mathbf{l_1, m_1, n_1}$ and $\mathbf{l_2, m_2, n_2}$.

Let θ be the angle between the given lines.

(*i*) $\sin^2\theta = 1-\cos^2\theta = 1-(l_1l_2+m_1m_2+n_1n_2)^2$. (Art. 1·16)

$$=(l_1^2+m_1^2+n_1^2)(l_2^2+m_2^2+n_2^2)-(l_1l_2+m_1m_2+n_1n_2)^2$$

(Note this step.)

$$(\because \ l_1^2+m_1^2+n_1^2=1=l_2^2+m_2^2+n_2^2)$$

$$=(m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2+(l_1m_2-l_2m_1)^2 \text{ (Art. 1·17)}$$

$$\therefore \ \mathbf{\sin\theta = \pm\sqrt{(m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2+(l_1m_2-l_2m_1)^2}}$$

$$\mathbf{= \pm\sqrt{\Sigma(m_1n_2-m_2n_1)^2}.}$$

(*ii*) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.

$$\therefore \ \mathbf{\tan\theta = \pm\sqrt{\Sigma(m_1n_2-m_2n_1)^2}/(l_1l_2+m_1m_2+n_1n_2).}$$

1·19. Condition of parallelism and perpendicularity.

To find the condition that the straight lines whose direction cosines are l_1, m_1, n_1 and l_2, m_2, n_2 are (i) parallel and (ii) perpendicular. (*Kashmir, 1951*)

Let θ be the angle between the lines.

(*i*) If the lines are parallel, θ is zero

$\therefore$ $\sin\theta = \sin 0 = 0$,

or $\sin^2\theta = 0$, or $(m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2+(l_1m_2-l_2m_1)^2=0$,

i.e., $m_1n_2-m_2n_1=0$, $\quad\therefore \ \dfrac{m_1}{m_2}=\dfrac{n_1}{n_2}$,

$n_1l_2-n_2l_1=0$, $\quad\therefore \ \dfrac{n_1}{n_2}=\dfrac{l_1}{l_2}$,

and $l_1m_2-l_2m_1=0$ $\quad\therefore \ \dfrac{l_1}{l_2}=\dfrac{m_1}{m_2}$.

Hence $\mathbf{\dfrac{l_1}{l_2}=\dfrac{m_1}{m_2}=\dfrac{n_1}{n_2}}$, which are the required conditions.

(*ii*) If the lines are perpendicular, θ is $90°$,

$\therefore$ $\cos\theta = \cos 90° = 0$.

Hence $\mathbf{l_1l_2+m_1m_2+n_1n_2=0}$, which is the required condition.

Note. *If the direction ratios of the two lines be a_1, b_1, c_1 and a_2, b_2, c_2 and the angle between them be θ,*

then $\cos\theta = (a_1a_2+b_1b_2+c_1c_2)/\sqrt{(a_1^2+b_1^2+c_1^2)(a_2^2+b_2^2+c_2^2)}$,

$\sin\theta = \pm\sqrt{(\ldots - \ldots}\ \sqrt{(a_1^2+b_1^2+c_1^2)(a_2^2+b_2^2+c_2^2)}$

and $\tan\theta = \pm\sqrt{\Sigma(b_1c_2-b \ldots}\ a_2+b_1b_2+c_1c_2)$.

Also, the conditions of parallelism and perpendicularity are respectively

$$\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2} \text{ and } a_1a_2+b_1b_2+c_1c_2=0.$$

The condition of parallelism as well as the condition of perpendicularity are of the same form for the two cases in which the direction cosines and direction ratios are used.

EXAMPLES I (C)

Type I. Ex. 1. Find the direction cosines of the lines joining the points (4, 3, −5) and (−2, 1, −8).

Sol. The direction ratios of the required line are

$$4+2,\ 3-1,\ -5+8, \text{ or, } 6,\ 2,\ 3.$$

∴ direction cosines of the line are

$$\frac{6}{\sqrt{36+4+9}},\ \frac{2}{\sqrt{36+4+9}},\ \frac{3}{\sqrt{36+4+9}}, \text{ or}$$

$$\frac{6}{7},\ \frac{2}{7},\ \frac{3}{7}.$$

Ex. 2. Find the direction cosines of the lines joining the following points :

(*i*) (4, 1, 3) and (1, 0, 2). (*ii*) (5, 7, 9), (3, 8, −4).

$$\left[\textbf{Ans. } \frac{3}{\sqrt{11}},\ \frac{1}{\sqrt{11}},\ \frac{1}{\sqrt{11}};\ \frac{2}{\sqrt{174}},\ \frac{-1}{\sqrt{174}},\ \frac{13}{\sqrt{174}}.\right]$$

Ex. 3. If P, Q, R are (2, 3, 5), (−1, 3, 2), (3, 5, −2), find the direction-cosines of the sides of the △ PQR. (*Patna, 1961 S*)

$$\left[\textbf{Ans. } -\frac{2}{3},\ \frac{-1}{3},\ \frac{2}{3};\ \frac{1}{3\sqrt{6}},\ \frac{2}{3\sqrt{6}},\ \frac{-7}{3\sqrt{6}};\ \frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}.\right]$$

Ex. 4. Prove that the three points P, Q, R whose coordinates are respectively (3, 2, −4), (5, 4, −6) and (9, 8, −10) are collinear and find the ratio in which Q divides PR. (*Bhagalpur, 1962 S*)

[**Ans.** 1 : 2.]

Type II. Ex. 1. If A, B, C, D are the points (3, 4, 5), (4, 6, 3), (−1, 2, 4) and (1, 0, 5), find the projection of CD on AB.

Sol. The direction ratios of CD are

$$1+1,\ 0-2,\ 5-4, \text{ or, } 2,\ -2,\ 1.$$

∴ The direction cosines of CD are

$$\frac{2}{\sqrt{4+4+1}},\ \frac{-2}{\sqrt{4+4+1}},\ \frac{1}{\sqrt{4+4+1}},$$

or, $$\frac{2}{3},\ \frac{-2}{3},\ \frac{1}{3}.$$

∴ from Art. 1·15, the projection of AB on CD

$$=(4-3)\left(\frac{2}{3}\right)+(6-4)\left(\frac{-2}{3}\right)+(3-5)\left(\frac{1}{3}\right)$$

$$=\frac{2}{3}-\frac{4}{8}-\frac{2}{3}=-\frac{4}{3}.$$

Ex. 2. If A, B, C, D are the points (6, −6, 0), (−1, −7, 6), (3, −4, 4), (2, −9, 2) respectively, prove that AB is perpendicular to CD.

Ex. 3. If A, B, C, D are the points (2, 3, −1), (5, 2, 3), (4, 3, −5) and (−2, 1, −8) respectively, find the projection of AB on CD.

[**Ans.** 4.]

Type III (A) Ex. 1. A variable line in two adjacent positions has direction cosines l, m, n; $l+\delta l$, $m+\delta m$, $n+\delta n$, show that the small angle $\delta\theta$ between the two positions is given by $(\delta\theta)^2=(\delta l)^2+(\delta m)^2+(\delta n)^2$.

(Agra, 1959 ; Punjab, 1954)

Sol. $\because$ l, m, n and $l+\delta l$, $m+\delta m$, $n+\delta n$ are direction-cosines,

$\therefore\ l^2+m^2+n^2=1$ and $(l+\delta l)^2+(m+\delta m)^2+(n+\delta n)^2=1$,

or, $\quad l^2+m^2+n^2=1$ and $(l^2+m^2+n^2)+[(\delta l)^2+(\delta m)^2+(\delta n)^2+2l\delta l+2m\delta m+2n\delta n]=1$,

or, $\quad l^2+m^2+n^2=1 \ldots(1)$ and $(\delta l)^2+(\delta m)^2+(\delta n)^2=-2(l\delta l+m\delta m+n\delta n) \ldots(2)$

Now, $\cos\delta\theta=l(l+\delta l)+m(m+\delta m)+n(n+\delta n)$. (Art. 1·16)

$$=l^2+m^2+n^2+l\delta l+m\delta m+n\delta n$$
$$=1-\tfrac{1}{2}[(\delta l)^2+(\delta m)^2+(\delta n)^2], \text{ using (1) and (2).}$$

or, $(\delta l)^2+(\delta m^2)+(\delta n)^2=2(1-\cos\delta\theta)=2.2\sin^2(\tfrac{1}{2}\delta\theta)$
$$=4(\tfrac{1}{2}\delta\theta)^2 \qquad (\because \sin\tfrac{1}{2}\delta\theta=\tfrac{1}{2}\delta\theta)$$
$$=(\delta\theta)^2.$$

Ex. 2. The coordinates of the angular points A, B, C, D of a tetrahedron are (−2, 1, 3), (3, −1, 2), (2, 4, −1) and (1, 2, 3) respectively. Find the angle between the edges AC and BD. *(Punjab, 1951 S)*

$$\left[\textbf{Ans. } \cos^{-1}\left(\frac{3}{\sqrt{374}}\right).\right]$$

Ex. 3. If P and Q are (2, 3, −6) and (3, −4, 5), find the angle that OP makes with OQ. *(Utkal, 1964)*

$$\left[\textbf{Ans. } \cos^{-1}\left(-\frac{18\sqrt{2}}{35}\right).\right]$$

(B) Ex. 1. A line makes angles α, β, γ, δ with the four diagonals of a cube; prove that $\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta=4/3$.

(A.M.I.E., 1961; Pakistan (Punjab), 1955; Vikram Engg., 1959; Karnatak, 1959; Delhi Hons., 1950; Patna, 1963; Magadh, 1963; Punjab B.Sc., 1964)

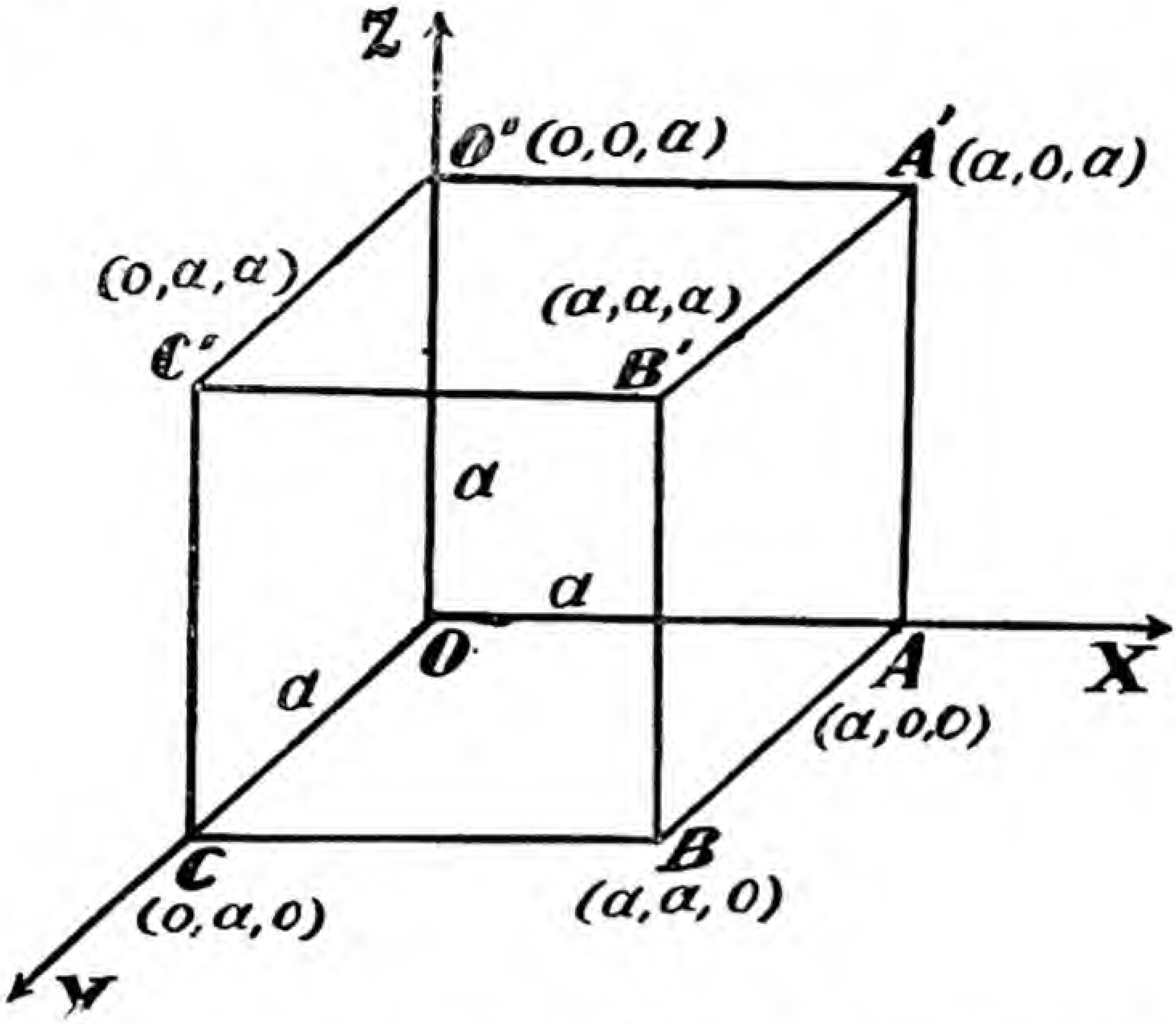
Sol. Let (OABC, O′A′B′C′) be a cube of side a.

Take O as the origin and the coterminous edges OA, OC, and OO′ as the axes.

$\therefore$ coordinates of A, B, C, A′, B′, C′, O′ are respectively (a, o, o), (a, a, o), (o, a, o), (a, o, a), (a, a, a), (o, a, a) and (o, o, a).

The four diagonals of the cube are AC′, OB′, CA′ and O′B.

The direction ratios of AC′, OB′, CA′, and O′B are respectively $-a, a, a$; a, a, a; $a, -a, a$ and $a, a, -a$.



∴ direction cosines of AC′, OB′, CA′ and O′B are respectively

$$-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}; \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}};$$
$$\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \text{ and } \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}.$$

Let l, m, n be the direction cosines of a line which makes angles $\alpha, \beta, \gamma, \delta$ with the diagonals AC′, OB′, CA′ and O′B.

$$\therefore \quad \cos\alpha=\frac{1}{\sqrt{3}}(-l+m+n), \ \cos\beta=\frac{1}{\sqrt{3}}(l+m+n),$$
$$\cos\gamma=\frac{1}{\sqrt{3}}(l-m+n), \ \cos\delta=\frac{1}{\sqrt{3}}(l+m-n).$$

$$\therefore \quad \cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta=\tfrac{1}{3}[(-l+m+n)^2+(l+m+n)^2+(l-m+n)^2+(l+m-n)^2]$$
$$=\tfrac{1}{3}[4(l^2+m^2+n^2)]$$
$$=\tfrac{4}{3} \quad (\because \ l^2+m^2+n^2=1).$$

Ex. 2. Find the angle between two diagonals of a cube.

(A.M.I.E., May, 1961 ; Gujarat, 1956 ; Punjab, 1964)

$$\left[\textbf{Ans.} \quad \cos^{-1}\frac{1}{\sqrt{3}}\right]$$

Ex. 3. If the edges of a rectangular parallelopiped are a, b, c, prove that the angles between the four diagonals are given by $\cos^{-1}\left[\dfrac{a^2\pm b^2\pm c^2}{a^2+b^2+c^2}\right]$

(Punjab, 1952 : Punjab (Pakistan), 1956 ; Bihar, 1963)

Type IV. Ex. 1. Show that the lines whose direction cosines are given by the equation $al+bm+cn=0$ and $ul^2+vm^2+wn^2=0$ are perpendicular or parallel according as $\Sigma a^2(v+w)=0$ or $\Sigma \frac{a^2}{u}=0$.

(Karnatak, 1961; Nagpur T.D.C., 1962; Punjab Hons., 1952 ; Punjab B.Sc., 1964)

Sol. The given equations are $al+bm+cn=0$ ...(1)

and $ul^2+vm^2+wn^2=0$ ...(2)

Eliminating n between (1) and (2) we have $ul^2+vm^2+w\left(-\frac{al+bm}{c}\right)^2=0$

or, $(c^2u+a^2w)l^2+2abwlm+(b^2w+c^2v)m^2=0$,

or, $(c^2u+a^2w)\left(\frac{l}{m}\right)^2+2abw.\frac{l}{m}+(b^2w+c^2v)=0$ ...(3)

Let (l_1, m_1, n_1) and (l_2, m_2, n_2) be the direction cosines of the lines.

$\therefore$ $\frac{l_1}{m_1}$ and $\frac{l_2}{m_2}$ are the roots of (3).

$\therefore$ $\frac{l_1l_2}{m_1m_2}$ = product of roots $=\frac{b^2w+c^2v}{c^2u+a^2w}$,

or, $\frac{l_1l_2}{b^2w+c^2v}=\frac{m_1m_2}{c^2u+a^2w}=\frac{n_1n_2}{a^2v+b^2u}$ by symmetry.

(*i*) The lines will be perpendicular,

if $l_1l_2+m_1m_2+n_1n_2=0$,

if $(b^2w+c^2v)+(c^2u+a^2w)+(a^2v+b^2u)=0$,

or if $a^2(v+w)+b^2(w+u)+c^2(u+v)=0$,

or if $\Sigma a^2(v+w)=0$.

(*ii*) The lines are parallel if $\frac{l_1}{m_1}=\frac{l_2}{m_2}$,

i.e., if the equation (3) has equal roots,

or if $a^2b^2w^2=(c^2u+a^2w)(b^2w+c^2v)$,

or if $a^2c^2vw+b^2c^2uw+c^4uv=0$,

or if $\frac{a^2}{u}+\frac{b^2}{v}+\frac{c^2}{w}=0$, on dividing both sides by uvw,

or if $\Sigma\frac{a^2}{u}=0$,

Ex. 2. Show that the straight lines whose direction cosines are given by the equations $ul+vm+wn=0$ and $fmn+gnl+hlm=0$, are perpendicular if $\frac{f}{u}+\frac{g}{v}+\frac{h}{w}=0$ and parallel if $\sqrt{uf}+\sqrt{vg}+\sqrt{wh}=0$.

Ex. 3. Show that the straight lines whose direction cosines are given by the equations $2l-m+2n=0$ and $mn+nl+lm=0$ are at right angles.

(Punjab B.Sc., 1965 S)

Ex. 4. Find the angle between the lines whose direction cosines are given by the equations $l+m+n=0$, $l^2+m^2-n^2=0$.

(A.M.I.E., Nov. 1955 ; Patna, 1959 ; Bihar, 1962 S ; Ranchi 1962 S)

$\left[\textbf{Ans. } \frac{\pi}{3} \text{ or } \frac{2\pi}{3}\right]$

Type V. (Condition of perpendicularity.)

Ex. 1. If two pairs of opposite edges of a tetrahedron be at right angles, then show that so is the third. *(Punjab, 1956S)*

Sol. Let O, ABC be a tetrahedron. Let O be the origin of reference.

Let the position vectors of A, B, C be **a**, **b** and **c**.

$\therefore\ \vec{AB}=\mathbf{b}-\mathbf{a},\ \vec{BC}=\mathbf{c}-\mathbf{b},\ \vec{CA}=\mathbf{a}-\mathbf{c}.$

$\because\quad OA\perp BC,$

$\therefore\quad \vec{OA}.\vec{BC}=0,$

or $\quad \mathbf{a}.(\mathbf{c}-\mathbf{b})=0,$

or $\quad \mathbf{a}.\mathbf{c}=\mathbf{a}.\mathbf{b}$ ...(1)

$\because\quad OB\perp CA$

$\therefore\quad \vec{OB}.\vec{CA}=0,$

or $\quad \mathbf{b}.(\mathbf{a}-\mathbf{c})=0,$

or $\quad \mathbf{b}.\mathbf{a}=\mathbf{b}.\mathbf{c}$ ...(2)

From (1) and (2), we have

$$\mathbf{b}.\mathbf{c}=\mathbf{a}.\mathbf{c}$$

or $\quad (\mathbf{b}-\mathbf{a}).\mathbf{c}=0$

$\quad \mathbf{c}.(\mathbf{b}-\mathbf{a})=0.$

$\therefore$ OC is perpendicular to AB.

Aliter. Let O, ABC be a tetrahedron. Let us take O as the origin. Let the coordinates of A, B, C are (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) respectively.

Let the opposite edges OA, BC and OB, CA be perpendicular.

It is required to show that the third pair OC and AB is also perpendicular.

The direction ratios of OA are x_1, y_1, z_1, that of BC are (x_3-x_2), (y_3-y_2) (z_3-z_2), that of OB are x_2, y_2, z_2, that of CA are (x_1-x_3), (y_1-y_3), (z_1-z_3) that of OC are x_3, y_3, z_3 and that of AB are $x_2-x_1, y_2-y_1, z_2-z_1$.

$\because\quad OA\perp BC,$

$\therefore\quad x_1 . (x_3-x_2)+y_1(y_3-y_2)+z_1(z_3-z_2)=0$ ...(1)

$\because\quad OB\perp CA$

$\therefore\quad x_2(x_1-x_3)+y_2(y_1-y_3)+z_2(z_1-z_3)=0$ ...(2)

Adding (1) and (2), we have

$x_3(x_1-x_2)+y_3(y_1-y_2)+z_3(z_1-z_2)=0$

This shows that OC and AB are also perpendicular.

Ex. 2. If two edges AB, CD of a tetrahedron ABCD are perpendicular, prove that the distance between the middle points of AC and BD is equal to the distance between the middle points of AD and BC.

Ex. 3. Prove that the three lines drawn from O with direction cosines l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3, are coplanar if

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}=0.$$

(Punjab, (Pakistan), 1954S ; Patna, 1960S ; Bhagalpur, 1962)

Sol. A line through O perpendicular to the plane containing the given lines is perpendicular to each of them. Let l, m, n be the direction cosines of that line.

$$\therefore \qquad ll_1+mm_1+nn_1=0 \qquad ...(1)$$

$$ll_2+mm_2+nn_2=0 \qquad ...(2)$$

and

$$ll_3+mm_3+nn_3=0 \qquad ...(3)$$

Eliminating l, m, n between (1), (2) and (3), we have

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}=0.$$

Ex. 4. Prove that the three lines drawn from the origin with direction cosines proportional to $1, -1, 1$; $2, -3, 0$; $1, 0, 3$ are coplanar.

(Calcutta, 1963)

Ex. 5. Lines OA and OB are drawn from the origin O with direction cosines proportional to $1, -2, -1$ and $3, -2, 3$. Find the direction cosines of the normal to the plane AOB. *(Baroda Engg., 1963)*

$$\left[\textbf{Ans.} \quad \frac{4}{\sqrt{29}}, \frac{3}{\sqrt{29}}, \frac{-2}{\sqrt{29}}\right]$$

MISCELLANEOUS EXAMPLES ON CHAPTER I

1. Find the direction cosines of a line that makes equal angles with the axes.

$$\left[\textbf{Ans.} \quad l=m=n=\pm\frac{1}{\sqrt{3}}.\right]$$

2. Prove that $\sin^2\alpha+\sin^2\beta+\sin^2\gamma=2$, in the usual notation.

3. The projections of line on the axes are 2, 3, 6. What is the length of the line ? **[Ans. 7]**

4. If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two mutually perpendicular lines, show that the direction cosines of the line perpendicular to them both are $m_1n_2-m_2n_1$, $n_1l_2-n_2l_1$, $l_1m_2-l_2m_1$.

5. If l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3 are the direction cosines of three mutually perpendicular lines OA, OB, OC, the line OP whose direction cosines are proportional to $l_1+l_2+l_3$, $m_1+m_2+m_3$, $n_1+n_2+n_3$ makes equal angles with them. *(Patna, 1960 ; Ranchi, 1963)*

6. There are three straight lines through the origin with direction cosines proportional to (1, 2, 2), (2, 3, 6), (3, 4, 12). Find the direction cosines of a straight line equally inclined to the three given lines. (*Patna, 1964*)

$$\left[\textbf{Ans. } -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}.\right]$$

7. Show that the lines whose direction cosines are given by $l+m+n=0$ and $2mn+3nl-5lm=0$ are perpendicular to each other. (*Magadh, 1964*)

8. Can the numbers $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$ be the direction cosines of any directed line? Give reasons for your answer. (*Calcutta, 1961*)

[**Ans.** No.]

2

Loci and their Equations

2·1. To find the locus of the equation f(x)=o.

Let the given equation be $f(x)=0$... (1), where y and z are absent.

Let (1) be expressed as $(x-\alpha_1)(x-\alpha_2)...=0$.

Now all the points whose coordinates satisfy (1) lie on one or other of the planes $x-\alpha_1=0$, $x-\alpha_2=0$,..., each plane being parallel to the yz-plane.

Hence **f(x)=o represents a system of planes parallel to the yz-plane.**

Note 1. *$f(y)=0$ represents a system of planes parallel to the zx-plane, and $f(z)=0$ represents a system of planes parallel to the xy-plane.*

Note 2. *These planes may be real or imaginary.*

2·2. To find the locus of the equation f(x, y)=o.

The given equation $f(x, y)=0$ ..(1) is the equation of a curve in the xy-plane.

Let A (α, β) be **any** point on this curve.

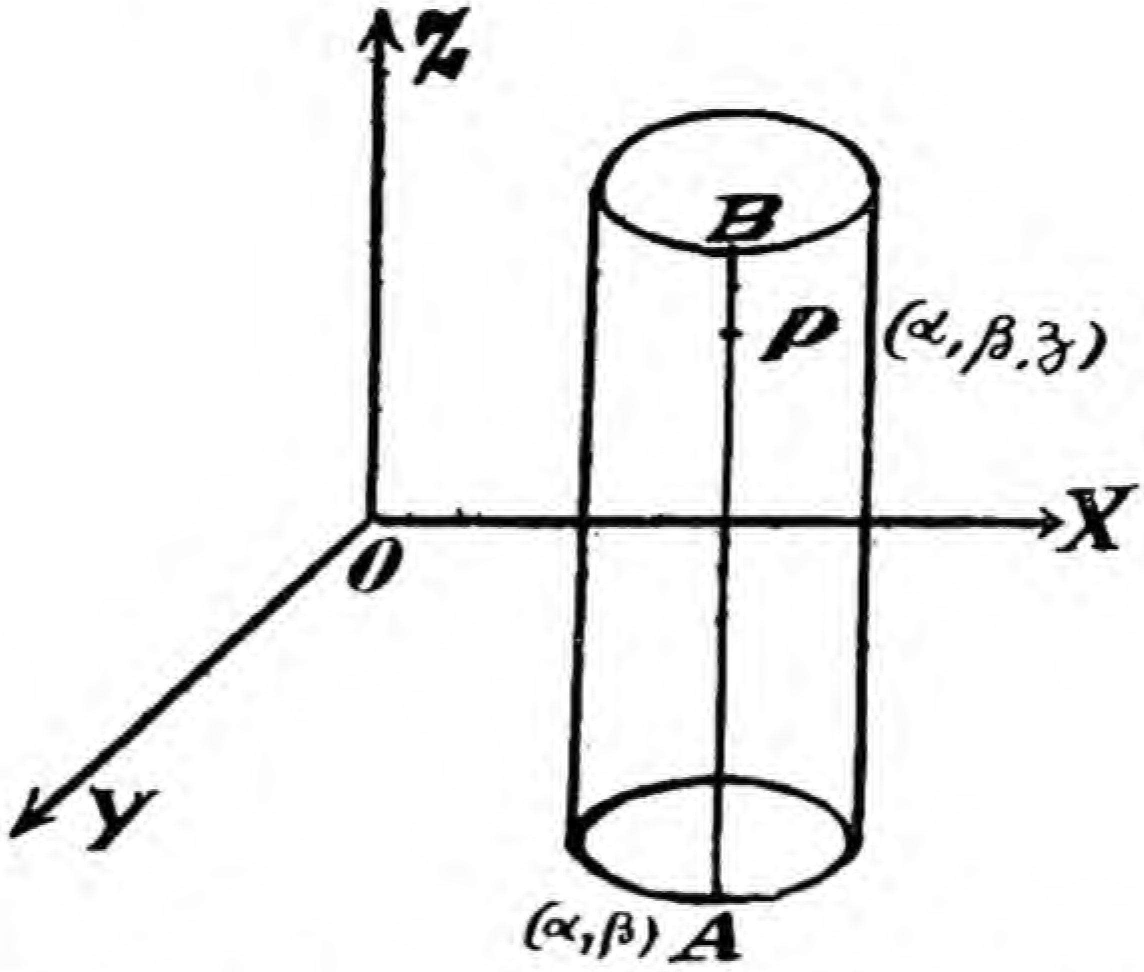
$$\therefore \qquad f(\alpha, \beta)=0 \qquad ...(2)$$

Through A draw a straight line AB parallel to the z-axis.

Let $P(\alpha, \beta, z)$ be **any** point on AB.

From (2), P lies on the locus of (1), *i.e.*, any point on the line through A parallel to the z-axis lies on the locus of (1), *i.e.*, the whole line lies on the locus of (1).

$\therefore$ the locus of (1) is the assemblage of lines, parallel to OZ drawn through the points on the curve $f(x, y)=0$.



Hence **f(x, y)=o represents a cylinder generated by a line which is drawn parallel to OZ and intersects the curve whose equation in the xy-plane is f (x, y)=o.**

Note. *$f(y, z)=0$ represents a cylinder whose generatars are parallel to the x-axis and which intersect the curve whose equation in the yz-plane is $f(y, z)=0$. Also, $f(z, x)=0$ represents a cylinder whose generators are parallel to the y-axis and which intersect the curve whose equation in the xz-plane is $f(z, x)=0$.*

2·3. To find the locus of the equation f(x, y, z)=o.

The given equation is $f(x, y, z)=0$ ...(1)

Let P be **any** point $(\alpha, \beta, 0)$ on the xy-plane.

Let, Q (α, β, z) be **any** point on the line through P parallel to OZ.

If Q lies on the locus of (1), then

$$f(\alpha, \beta, z)=0.$$

This equation in z has a finite number of roots.

$\therefore$ the line through any point P in the xy-plane parallel to OZ cuts the locus of (1) in a finite number of points.

$\therefore$ the locus, which is the assemblage of all such points for different values of α, β, is a surface and not a solid.

Hence **f(x, y, z)=o represents a surface and not a solid.**

2·4. Equation of a curve. To find the locus of

$$\mathbf{f(x, y, z)=0, \quad F(x, y, z)=0.}$$

The given equations are

$$f(x, y, z)=0, F(x, y, z)=0 \qquad ...(1)$$

The points, whose coordinates satisfy these equations simultaneously, are common to the two surfaces separately represented by them.

$\therefore$ these points lie on the curve of intersection of these surfaces.

Hence (1) **represents the curve of intersection of the surfaces f(x, y, z) = 0 and F(x, y, z) = 0.**

Cor. A curve in the xy-plane.

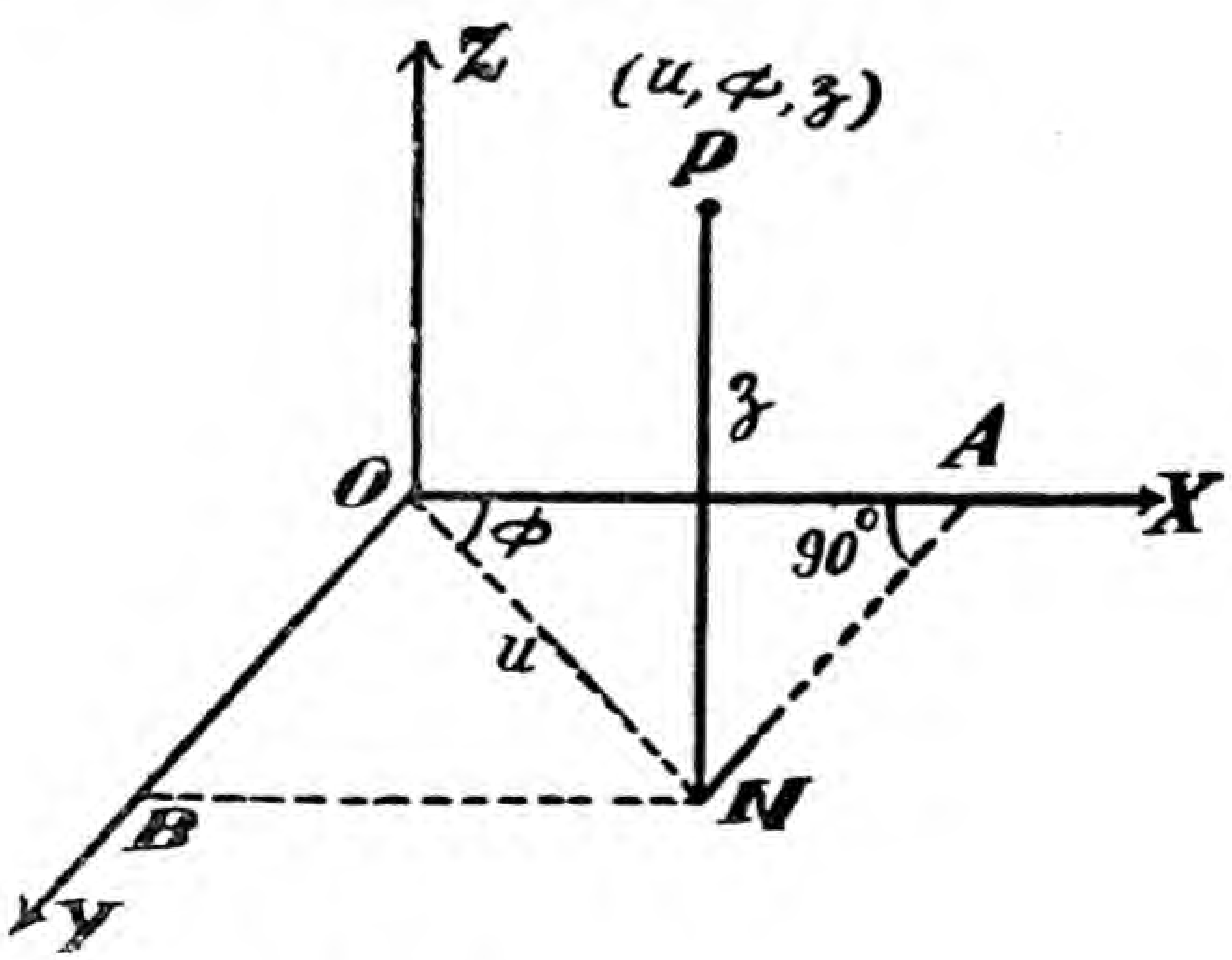
$\because$ the curve in the xy-plane is the curve of intersection of the cylinder $f(x, y)=0$ and the xy-plane $z=0$,

$\therefore$ the **equations of the curve in the xy-plane are**

$$\mathbf{f(x, y)=0, \quad z=0.}$$

2·5. Cylindrical coordinates of a point in space.

Let P be any point in space. Draw PN perpendicular to the xy-plane. $\therefore$ $PN=z$ is the z-coordinate of P.



Join ON. Let ON be u.

Let ON make an angle ϕ with OX. Draw NA and NB respectively at right angles to OX and OY.

$$\therefore OA=x=u \cos \phi, \quad OB=y=u \sin \phi.$$

$\therefore$ **The cylindrical coordinates of P are $(\mathbf{u}, \phi, \mathbf{z})$, where**

$$\mathbf{u} = \sqrt{\mathbf{x}^2+\mathbf{y}^2} \quad \text{and} \quad \phi = \tan^{-1}\frac{\mathbf{y}}{\mathbf{x}}.$$

Note. *(i) z is regarded as positive if P lies above the xy-plane on the side of OZ and is regarded as negative if P is below the xy-plane on the side OZ'.*

(ii) u is essentially positive.

(iii) ϕ varies from 0 to 2π.

2·6. (Surface of revolution.) To find the surface formed by rotating the curve $f(y, z)=0$ about the z-axis.

Let $f(y, z)=0$...(1) be the equation of the curve in the yz-plane. Let $P(0, \beta, \gamma)$ be **any** point on (1).

$$\therefore \qquad f(\beta, \gamma)=0 \qquad \ldots(2)$$

The rotation of (1) about oz produces a surface of revolution.

As P moves round the surface, γ, the z-coordinate of P remains unchanged, and u, the distance of P from the z-axis, is always equal to β.

$\therefore$ from (1), the cylindrical coordinates of P satisfy the equation $f(u, z)=0$.

$\therefore$ P is **any** point on the curve, or surface,

$\therefore$ the cylindrical equation to the surface is $f(u, z)=0$.

Hence **the cartesian equation of the surface of revolution formed by rotating the curve $f(y, z)=0$ about the z-axis is**

$$\mathbf{f}(\sqrt{\mathbf{x}^2+\mathbf{y}^2},\ \mathbf{z})=0.$$

Note. *Equation of the form $f(\sqrt{y^2+z^2}, x)=0$ represents a surface of revolution whose axis is OX, and $f(\sqrt{z^2+x^2}, y)=0$ represents a surface of revolution whose axis is OY.*

Ex. *Find the equation of the surface of revolution generated by the rotation of the curve $y=f(z)$, $x=0$, about the z-axis.* *(Delhi Engg., 1962)*

EXAMPLES II

Type I. Ex. 1. Discuss the form of the surface represented by :

$$\mathbf{x}^2+\mathbf{y}^2=2\mathbf{az}.$$

Sol. The given equation is $x^2+y^2=2az$ ...(1)

The section of (1) by the plane $z=c$ is

$$x^2+y^2=2ac,\ z=c.$$

This represents a real circle if $c>0$. It reduces to a point if $c=0$.

The section of (1) by the plane $y=d$ is

$$x^2+d^2=2az,\ y=d,$$

which represents a real parabola for all real values of d.

Also, the section of (1) by the plane $x=e$ is

$$y^2+e^2=2az,\ x=e,$$

which represents a real parabola for all real values of e. This surface is called the paraboloid of revolution.

Ex. 2. Discuss the form of the following surfaces represented by

(i) $x^2+y^2+z^2=a^2$,

(ii) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$

(iii) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1$

(iv) $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1$.

Type II. Ex. 1. Find the equations of the cylinders with generators parallel to OX, OY, OZ, which pass through the curve of intersection of the surfaces represented by $x^2+y^2+2z^2=12$, $x-y+z=1$.

Sol. To obtain the equation of the cylinder with generators parallel to the x-axis, we shall eliminate x between the given surfaces.

$\therefore$ we have, $\quad (1+y-z)^2+y^2+2z^2=12$,

or, $\quad 2y^2+3z^2-2yz+2y-2z-11=0.$

Eliminating y between the equations of the given surfaces, the equation of the cylinder with generators parallel to the y-axis is

$$x^2+(x+z-1)+2z^2=12,$$

or, $\quad 2x^2+3z^2+2xy-2x-2z-11=0.$

Again eliminating z between the equations of the given surfaces, the equation of the cylinder with generators parallel to the z-axis is

$$x^2+y^2+2(1-x+y)^2=12.$$

or, $\quad 3x^2+3y^2-4xy-4x+4y-10=0.$

Ex. 2. Find what loci are represented by

(ii) $x^2+y^2=a^2,\ x^2=b^2,\ (a>b^2)$,

(ii) $x^2+y^2+z^2=a^2,\ x^2+y^2=2az$.

Type III. Ex. 1. Find the equation to the cone formed by rotating the line $z=0$, $y=2x$ about OX. (*Punjab B.Sc., 1958S*)

Sol. From the note of Art. 2·6, the equation of the surface of revolution having the axis OX is of the form

$$f(\sqrt{y^2+z^2},\ x)=0.$$

The given line is $z=0$, $y=2x$.

$\therefore$ the equation of the required cone is

$$y^2+z^2=4x^2,$$

or,

$$4x^2-y^2-z^2=0.$$

Ex. 2. Show that the surface generated by rotating the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z=0$$

about its axes are given by

$$\frac{x^2}{a^2}+\frac{y^2+z^2}{b^2}=1,\quad \frac{x^2+z^2}{a^2}+\frac{y^2}{b^2}=1.$$

Ex. 3. Find the equation to the surface generated by the revolution of the circle $x^2+y^2+2ax+b^2=0$, $z=0$, about the y-axis.

[**Ans.** $(x^2+y^2+z^2+b^2)^2=4a^2(x^2+z^2)$.]

3

The Plane

3·1. The plane : Def.

A plane is a surface such that if any two points be taken on it, the line joining them lies wholly in the surface.

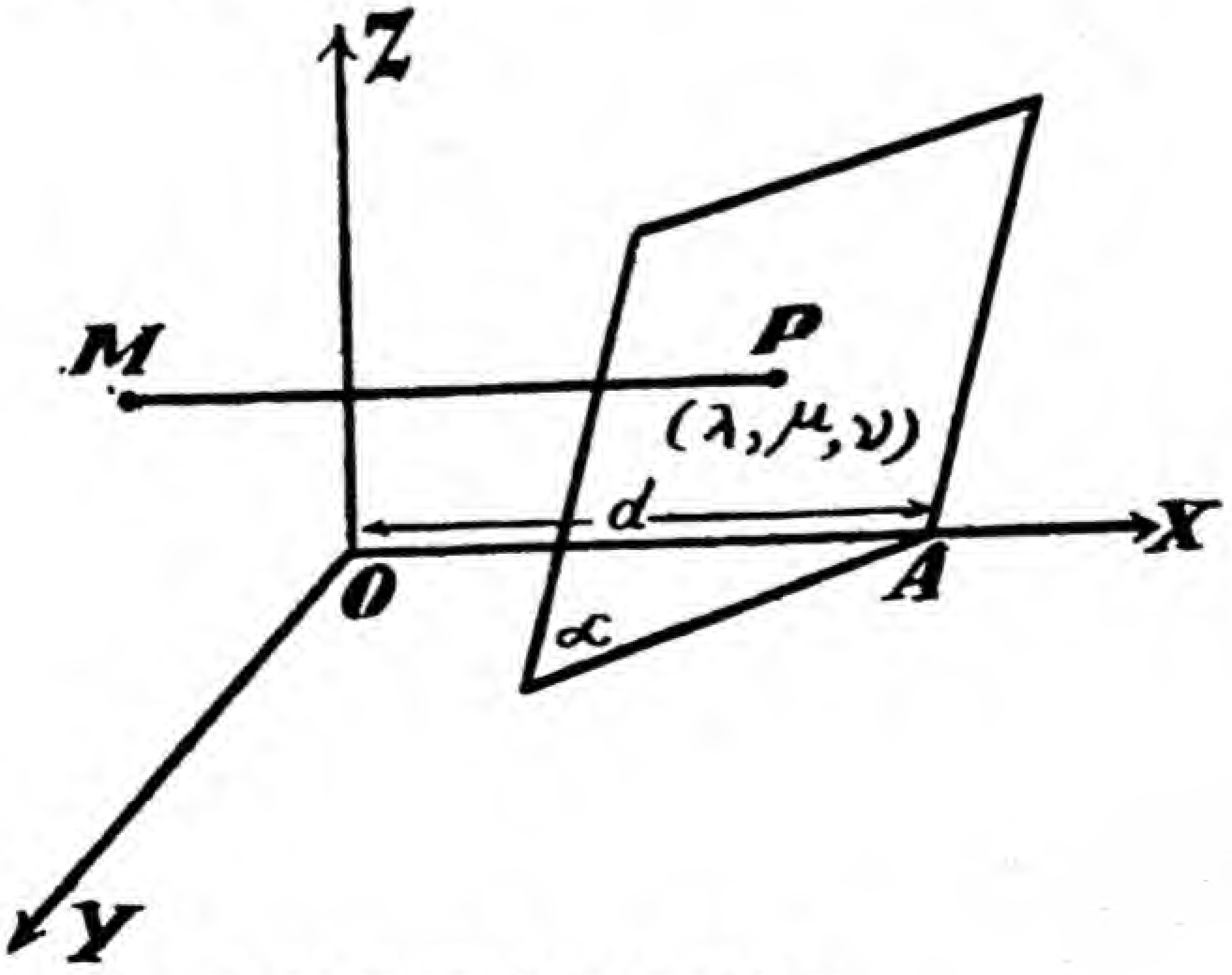
SECTION I

Forms of Equations of Planes

3·2. Plane parallel to the yz-plane.

To find the equation of the plane parallel to the yz-plane and at a distance d from it.

Let P (λ, μ, ν) be any point on the plane α meeting the x-axis in A, so that $OA=d$.



Draw PM perpendicular to the yz-plane.

$\therefore$ $$MP=OA,$$

or, $$\lambda=d.$$

Hence the locus of (λ, μ, ν) is $\mathbf{x=d}$, which is the required equation.

Cor. Equation of the yz-plane.

In this case $d=0$.

$\therefore$ equation of the yz-plane is $\mathbf{x=o}$.

Note 1. *The equations of the planes parallel to the zx-plane and xy-plane and at distances e and f respectively from them are* $\mathbf{y=e}$ **and** $\mathbf{z=f}$.

Note 2. *Equations of the zx-plane and xy-plane are respectively* $\mathbf{y=0}$, $\mathbf{z=0}$.

3·3. General equation of the first degree in x, y, z : Def.

An equation of the form $Ax+By+Cz+D=0$ is called the general equation of the first degree in x, y and z.

3·4. General form of the equation of the plane.

To prove that the general equation of the first degree in x, y and z always represents a plane.

Proof. Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be **any** two points on the locus represented by the equation $Ax+By+Cz+D=0$ ...(1)

$\because$ P lies on (1), $\therefore$ $Ax_1+By_1+Cz_1+D=0$ ...(2)

$\because$ Q lies on (1), $\therefore$ $Ax_2+By_2+Cz_2+D=0$ ...(3)

Multiplying (3) λ and adding to (2), we have

$$A\left(\frac{x_1+\lambda x_2}{1+\lambda}\right)+B\left(\frac{y_1+\lambda y_2}{1+\lambda}\right)+C\left(\frac{z_1+\lambda z_2}{1+\lambda}\right)+D=0 \quad ...(4)$$

This relation shows that the point

$$\left(\frac{x_1+\lambda x_2}{1+\lambda}, \frac{y_1+\lambda y_2}{1+\lambda}, \frac{z_1+\lambda z_2}{1+\lambda}\right)$$

which divides the line PQ in the ratio $\lambda : 1$, also lies on the locus of (1).

$\because$ λ can have any value,

$\therefore$ it follows that the co ordinates of all the points on the line PQ satisfy the locus of (1), *i.e.*, the line PQ, joining any two arbitrary points P and Q on the locus of (1), lies wholly on the locus of (1).

$\therefore$ by def., Art 3·1, the locus of (1) must be a plane.

This proves the proposition.

Note 1. Why is it called the general form ?

$Ax+By+Cz+D=0$ is called the general form of the equation of the plane, because by proper selection of A, B, C, D it can be made to represent the equation of the plane in any form.

Note 2. Number of constans in $Ax+By+Cz+D=0$.

The equation $Ax+By+Cz+D=0$ can be written as

$$\frac{A}{D}x+\frac{B}{D}y+\frac{C}{D}z+1=0.$$

or,

$$A_1x+B_1y+C_1z+1=0,$$

where $A_1\equiv\frac{A}{D}$, $B_1\equiv\frac{B}{D}$, $C_1\equiv\frac{C}{D}$.

This shows that although the above equation contains four constants A, B, C, D, but in reality it contains **only three independent constants.**

Note 3. *The plane passing through the origin is $Ax+By+Cz=0$, because in this case $D=0$.*

3·5. One-point form of the equation of the plane. To find the equation of the plane through the point (x_1, y_1, z_1).

Let $$Ax+By+Cz+D=0 \qquad ...(1)$$
be the required plane.

$\because$ (1) passes through $(x_1\ y_1, z_1)$,

$\therefore$ $$Ax_1+By_1+Cz_1+D=0 \qquad ...(2)$$

Subtracting (2) from (1), we have
$$A(x-x_1)+B(y-y_1)+C(z-z_1)=0.$$

Hence the required equation is

$$\mathbf{A(x-x_1)+B(y-y_1)+C(z-z_1)=0.}$$

3·6. Perpendicular form of the equation of the plane. To find the equation of a plane in terms of the length of the perpendicular p from the origin upon it and l, m, n, the direction-cosines of this perpendicular.

Let ABC be the plane.

Let $OM(=p)$ be the perpendicular from O on the plane ABC.

Let l, m, n be the direction cosines of OM. Let $\hat{\mathbf{e}}$ be the unit vector along OM.

Let $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ be the unit vectors along the positive axes of x, y and z of a right-handed system.

Let $P(x, y, z)$ be **any** point on the plane ABC.

Let $\overrightarrow{OP}$ be $\mathbf{r}$ and OP be r.

$\therefore \qquad \mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}.$

Also, $\qquad \hat{\mathbf{e}}=l\mathbf{i}+m\mathbf{j}+n\mathbf{k}$

$\therefore \qquad \mathbf{r}.\hat{\mathbf{e}}=lx+my+nz \qquad ...(1)$

But $\qquad \mathbf{r}.\hat{\mathbf{e}}=r.1.\cos POM$, (by def. of a scalar product of two vectors.)

$\qquad =OP \cos POM=OM=p. \qquad ...(2)$

From (1) and (2), we have

$lx+my+nz=p$, which is the required equation.

Note. The vector form of the above equation is $\mathbf{r}.\hat{\mathbf{e}}=p$.

If there by any vector $\mathbf{e}$ parallel to $\hat{\mathbf{e}}$ and modulus e,

then $\qquad \mathbf{e}=e\hat{\mathbf{e}}.$

Multiplying both sides of $\mathbf{r}.\hat{\mathbf{e}}=p$ by e.

We have $\qquad e(\mathbf{r}.\hat{\mathbf{e}})=ep,$

or, $\qquad \mathbf{r}.\mathbf{e}=ep=q$, where $p=q/e$.

$\therefore$ $\mathbf{r}.\mathbf{e}=q$ represents a plane, the length of the perpendicular from the origin on it is obtained by dividing the R.H.S. by the modulus of $\mathbf{e}$.

$\therefore$ if a plane be $7x+9y+3z=11$,
then the corresponding vector equation is

$$(x\mathbf{i}+y\mathbf{j}+z\mathbf{k}).(7\mathbf{i}+9\mathbf{j}+3\mathbf{k})=11,$$

and the length of the perpendicular from the origin on it

$$=\frac{11}{\sqrt{7^2+9^2+3^2}}=\frac{11}{\sqrt{139}}.$$

Aliter. Let ABC be the plane and $OM=p$ be the perpendicular from O on it.

Let l, m, n be the direction cosines of OM.

Let $P(\lambda, \mu, \nu)$ be **any** point on the plane. Join OP and PM.

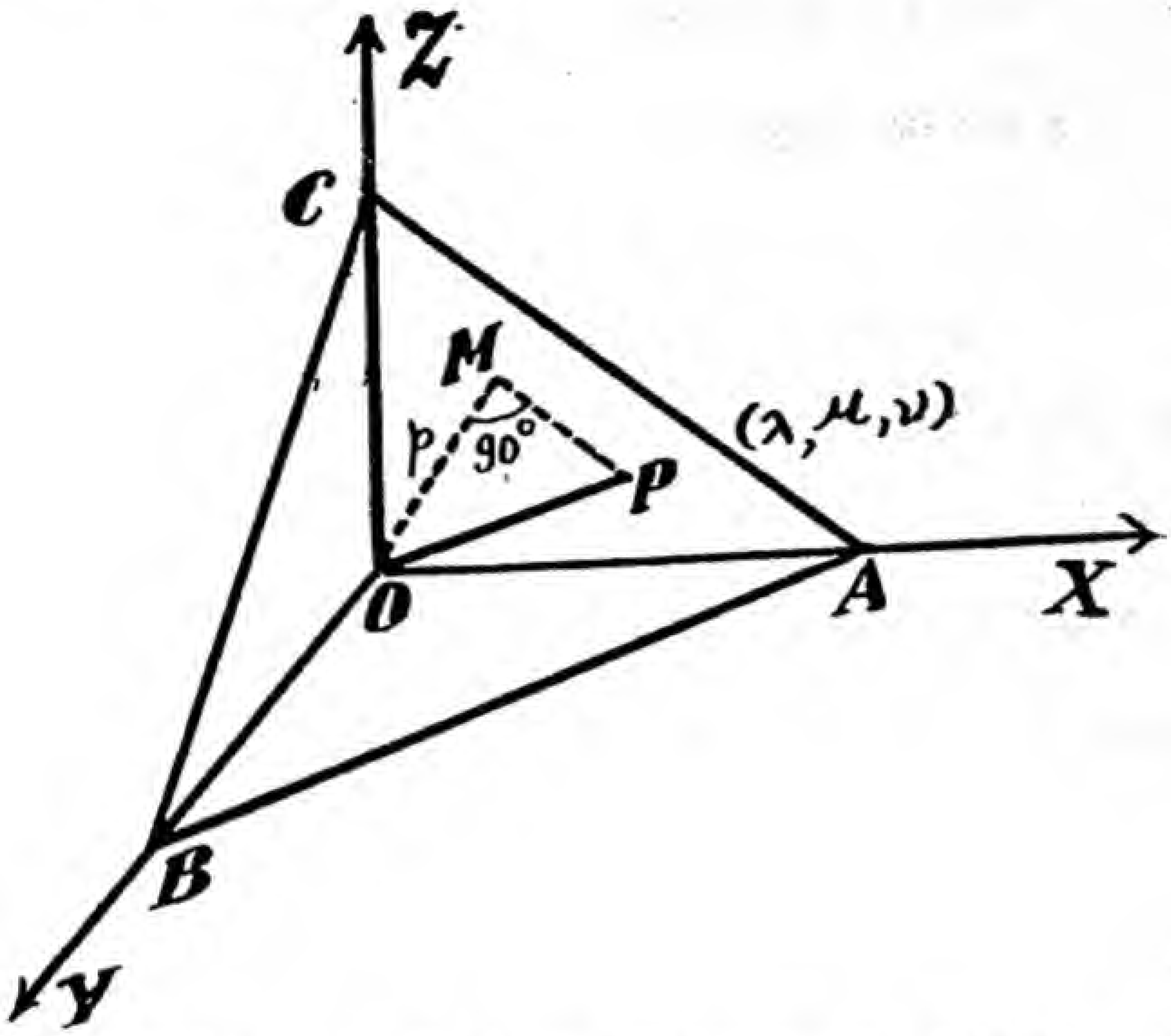
$\because$ OM is a line perpendicular to the plane and PM is a line lying in the plane,

$\therefore \qquad \angle OMP=90°.$

Now, OM=projection of OP on OM, a line having direction-cosines l, m, n

or $$p=(\lambda-0)l+(\mu-0)m+(\nu-0)n,$$

or $$l\lambda+m\mu+n\nu=p.$$



$\therefore$ locus (λ, μ, ν) is $\mathbf{lx+my+nz=p}$, which is the required equation.

Note 1. *Here l, m, n are the actual direction cosines of the normal to the plane and p is always regarded as positive.*

Note 2. Why it is called the perpendicular form ?

It is called the perpendicular form of the equation of the plane because it involves the length of the perpendicular from the origin to the plane.

Note 3. *The point, where the perpendicular from the origin on the plane meets the plane, is* $(\mathbf{lp, mp, np})$.

3·7. Triple intercept form of the equation of the plane. To find the equation of a plane which cuts off intercepts a, b and c respectively from the axes of x, y and z.

Let ABC be the plane cutting the axes in A, B, C respectively.

Let $OA=a$, $OB=b$ and $OC=c$.

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be the position vectors of the points A, B, C.

Let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be the unit vectors along the positive axes of x, y and z of a right-handed system.

$$\therefore \quad \vec{OA}=a\mathbf{i}, \quad \vec{AB}=b\mathbf{j}, \quad \vec{OC}=c\mathbf{k},$$

or, $\quad \mathbf{a}=a\mathbf{i}, \quad \mathbf{b}=b\mathbf{j}, \quad \mathbf{c}=c\mathbf{k}.$

$$\therefore \quad \vec{AB}=b\mathbf{j}-a\mathbf{i} \text{ and } \vec{AC}=c\mathbf{k}-a\mathbf{i}.$$

$\therefore$ the required plane is a plane passing through A and parrallel to the vectors $(b\mathbf{j}-a\mathbf{i})$ and $(c\mathbf{k}-a\mathbf{i})$.

$\therefore$ equation of the plane is

$$\mathbf{r}=a\mathbf{i}+s(b\mathbf{j}-a\mathbf{i})+t(c\mathbf{k}-a\mathbf{i}),^*$$

where $\mathbf{r}$ is the position vector of **any** point P on the plane whose cartesian coordinates are (x, y, z) and s, t are variable scalars which vary as the point P moves on the plane,

or, $\quad x\mathbf{i}+y\mathbf{j}+z\mathbf{k}=(1-s-t)a\mathbf{i}+sb\mathbf{j}+tc\mathbf{k}.$

Equating the coefficients of like vectors, we have

$$x=(1-s-t)a, \; y=sb, \; z=tc,$$

or, $\quad \dfrac{x}{a}=1-s-t, \; \dfrac{y}{b}=s, \; \dfrac{z}{c}=t.$

Adding these results, we have

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1,$$

which is the required equation.

Aliter. Now points P, A, B, C lie in the same plane.

$\therefore$ vectors $\vec{AP}$, $\vec{AB}$, $\vec{BC}$ are coplanar.

$\therefore$ their scalar triple product is zero.

$$\therefore \quad \vec{AP}.\,(\vec{AB}\times\vec{BC})=0,$$

or, $\quad (\mathbf{r}-a\mathbf{i}).[(b\mathbf{j}-a\mathbf{i})\times(c\mathbf{k}-b\mathbf{j})]=0,$ 16470.

or, $\quad (x\mathbf{i}+y\mathbf{j}+z\mathbf{k}-a\mathbf{i}).[bc\mathbf{i}+ac\mathbf{j}+ab\mathbf{k}]=0$

or, $\quad xbc+yac+abz-abc=0,$

or, $\quad \dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=1.$

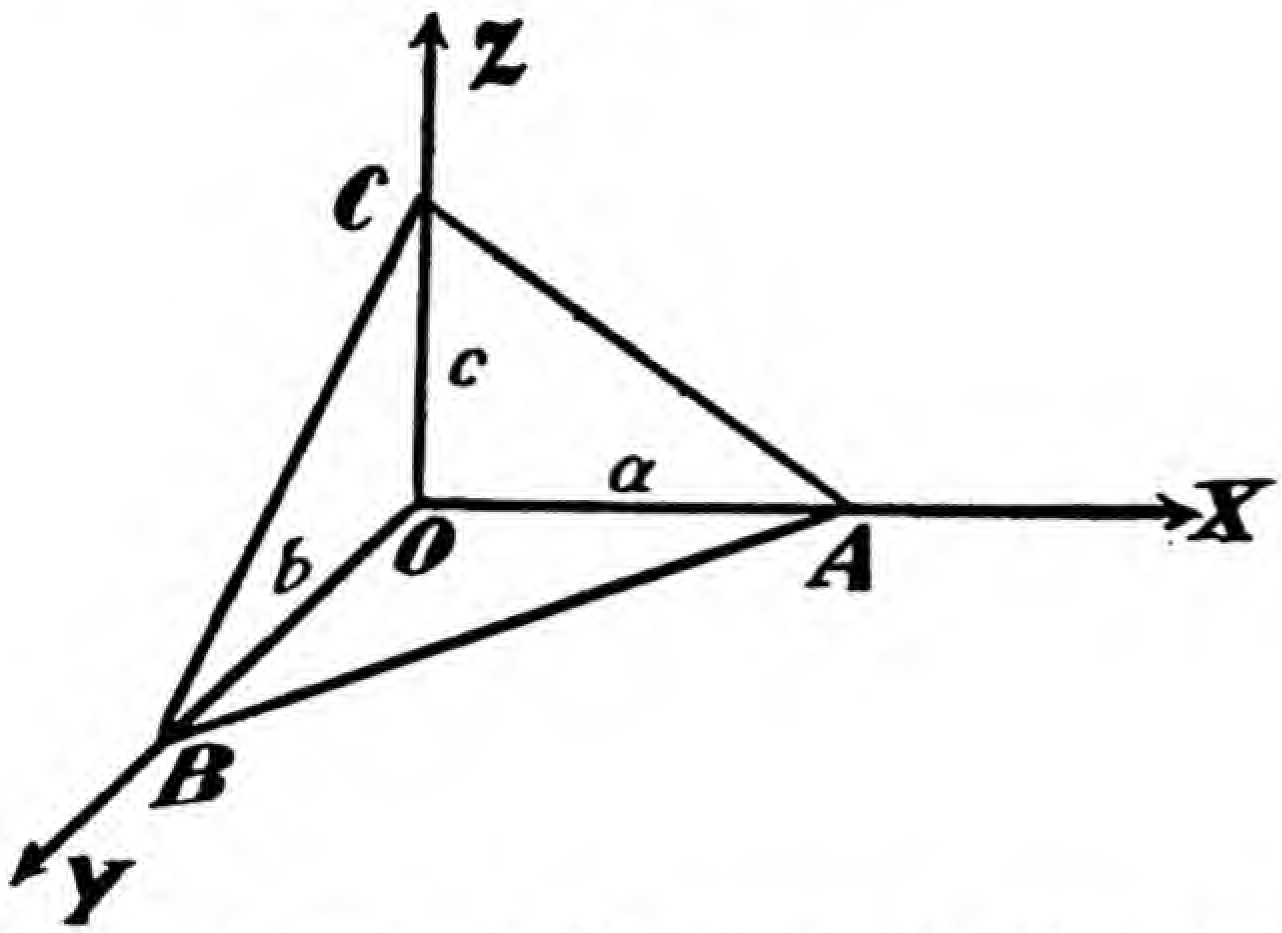
Aliter. Let ABC be the plane meeting the axes in A, B and C respectively. Then $OA=a$, $OB=b$ and $OC=c$. The coordinates of A, B, C are respectively $(a, o, o,)$, (o, b, o) and (o, o, c).

---

*See Author's *New Text book of Vector Algebra*, Art. 2·11.

Let the equation of the plane ABC be

$$Ax+By+Cz+D=0 \qquad \ldots(1)$$



$\because$ it passes through A, $Aa+D=0$, or, $A=-D/a$.
$\because$ it passes through B, $Bb+D=0$, or, $B=-D/b$.
$\because$ it passes through C, $Cc+D=0$, or, $C=-D/c$.

$\therefore$ (1) becomes $-\dfrac{D}{a}x-\dfrac{D}{b}y-\dfrac{D}{c}z+D=0$,

or $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=1$, which is the required equation.

3·8 Three-point form of the equation of a plane.

To find the equation of the plane through three points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3).

Let **a, b, c** be the position vectors of the points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ respectively.

Let **i, j, k** be the unit vectors in the positive directions of the axes of x, y and z of a right-handed system.

$\therefore$ $\mathbf{a}=x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k}$, $\quad \mathbf{b}=x_2\mathbf{i}+y_2\mathbf{j}+z_2\mathbf{k}$,
$\mathbf{c}=x_3\mathbf{i}+y_3\mathbf{j}+z_3\mathbf{k}$.

$\therefore$ $\overrightarrow{AB}=\mathbf{b}-\mathbf{a}$ and $\overrightarrow{AC}=\mathbf{c}-\mathbf{a}$.

The required plane is a plane through the point **a** and parallel to the vectors $(\mathbf{b}-\mathbf{a})$ and $(\mathbf{c}-\mathbf{a})$.

$\therefore$ its equation is

$$\mathbf{r}=\mathbf{a}+s(\mathbf{b}-\mathbf{a})+t(\mathbf{c}-\mathbf{a}),$$

where **r** is the position vector of any point P on the plane, whose cartesian co-ordinates are (x, y, z) and s, t are the scalars.

or, $$\mathbf{r}=(1-s-t)\mathbf{a}+s\mathbf{b}+t\mathbf{c} \qquad ...(\mathbf{A}),$$

or, $$(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})=(1-s-t)(x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k})$$
$$+s(x_2\mathbf{i}+y_2\mathbf{j}+z_2\mathbf{k})$$
$$+t(x_3\mathbf{i}+y_3\mathbf{j}+z_3\mathbf{k}).$$

Equating coefficients of like vectors, we have

$$(x-x_1)-s(x_2-x_1)-t(x_3-x_1)=0,$$
$$(y-y_1)-s(y_2-y_1)-t(y_3-y_1)=0$$

and $$(z-z_1)-s(z_2-z_1)-t(z_3-z_1)=0.$$

Eliminating $-s$ and $-t$, we have

$$\begin{vmatrix} x-x_1 & x_2-x_1 & x_3-x_1 \\ y-y_1 & y_2-y_1 & y_3-y_1 \\ z-z_1 & z_2-z_1 & z_3-z_1 \end{vmatrix}=0,$$

or, $$\begin{vmatrix} x-x_1 & x_2-x_1 & x_3-x_1 & x_1 \\ y-y_1 & y_2-y_1 & y_3-y_1 & y_1 \\ z-z_1 & z_2-z_1 & z_3-z_1 & z_1 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Adding fourth column to first, second and third columns, we have

$$\begin{vmatrix} x & x_2 & x_3 & x_1 \\ y & y_2 & y_3 & y_1 \\ z & z_2 & z_3 & z_1 \\ 1 & 1 & 1 & 1 \end{vmatrix}=0, \quad \text{or,} \quad \begin{vmatrix} x & y & z & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_1 & y_1 & z_1 & 1 \end{vmatrix}=0,$$

or, $$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix}=0,$$

which is the required equation.

Aliter. Let $Ax+By+Cz+D=0$ ...(1)
be the required plane.

$\because$ it passes through (x_1, y_1, z_1),

$\therefore$ $Ax_1+By_1+Cz_1+D=0$ ...(2)

$\because$ it passes through (x_2, y_2, z_2),

$\therefore$ $Ax_2+By_2+Cz_2+D=0$ ...(3)

$\because$ it passes through (x_3, y_3, z_3),

$\therefore$ $Ax_3+By_3+Cz_3+D=0$ ...(4)

Eliminating A, B, C, D between (1), (2), (3) and (4), we have

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix}=0,$$

which is the required equation of the plane.

Cor. Condition of coplanarity of four points.

The condition that the four points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) are coplanar is

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0.$$

REDUCTION OF GENERAL EQUATION INTO TRIPLE INTERCEPT FORM AND PERPENDICULAR FORM

3·9. To reduce the equation $\mathbf{Ax+By+Cz+D=0}$ to the form (i) $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=1$ (ii) $\mathbf{lx+my+nz=p}$.

(*i*) The given equation is

$$Ax+By+Cz+D=0 \qquad \text{...(1)}$$

Transposition gives

$$Ax+By+Cz=-D,$$

or,

$$-\frac{A}{D}x-\frac{B}{D}y-\frac{C}{D}z=1,$$

or,

$$\frac{x}{(-A/D)}+\frac{y}{(-B/D)}+\frac{z}{(-C/D)}=1,$$

which is required triple intercept form.

(*ii*) The given equation is

$$Ax+By+Cz+D=0 \qquad \text{...(1)}$$

If it is the same as

$$lx+my+nz=p,$$

or

$$lx+my+nz-p=0, \qquad \text{...(2)}$$

then comparing coefficients of like terms in (1) and (2),

$$\frac{l}{A}=\frac{m}{B}=\frac{n}{C}=\frac{-p}{D}.$$

Now, each $= \dfrac{\sqrt{l^2+m^2+n^2}}{\pm\sqrt{A^2+B^2+C^2}} = \dfrac{1}{\pm\sqrt{A^2+B^2+C^2}}.$

$$\therefore \quad l=\frac{A}{\pm\sqrt{A^2+B^2+C^2}}; \quad m=\frac{B}{\pm\sqrt{A^2+B^2+C^2}},$$

$$n=\frac{C}{\pm\sqrt{A^2+B^2+C^2}} \qquad \text{...(3)}$$

and $$p=\frac{-D}{\pm\sqrt{A^2+B^2+C^2}}. \qquad \ldots(4)$$

Case I. When D is positive.

$\because$ p is always positive,

$\therefore$ from (4), the sign of the radical must be negative.

$\therefore$ from (3), $l=\dfrac{-A}{\sqrt{A^2+B^2+C^2}}$, $m=\dfrac{-B}{\sqrt{A^2+B^2+C^2}}$,

$$n=\frac{-C}{\sqrt{A^2+B^2+C^2}},$$

and from (4), $$p=\frac{-D}{-\sqrt{A^2+B^2+C^2}}=\frac{D}{\sqrt{A^2+B^2+C^2}}$$

$\therefore$ (2) becomes

$$-\frac{A}{\sqrt{A^2+B^2+C^2}}x-\frac{B}{\sqrt{A^2+B^2+C^2}}y-\frac{C}{\sqrt{A^2+B^2+C^2}}z-\frac{D}{\sqrt{A^2+B^2+C^2}},$$

which is the required perpendicular form.

Case II. When D is negative.

$\because$ p is always positive,

$\therefore$ from (4), the sign of the radical must be positive.

$\therefore$ from (3),

$$l=\frac{A}{\sqrt{A^2+B^2+C^2}},\quad m=\frac{B}{\sqrt{A^2+B^2+C^2}},$$

$$n=\frac{C}{\sqrt{A^2+B^2+C^2}}$$

and from (4), $$p=\frac{-D}{\sqrt{A^2+B^2+C^2}}.$$

$\therefore$ (2) becomes

$$\frac{A}{\sqrt{A^2+B^2+C^2}}x+\frac{B}{\sqrt{A^2+B^2+C^2}}y+\frac{C}{\sqrt{A^2+B^2+C^2}}z=-\frac{D}{\sqrt{A^2+B^2+C^2}}$$

which is the required perpendicular form.

Note 1. Direction ratios of the normal to the plane.

(Karnatak Engg., 1961)

If $Ax+By+Cz+D=0$ be the equation of the plane, then **A, B, C** *are the direction ratios of the normal to the plane.*

Note 2. *The R.H.S. in the above Art. should invariably be positive.*

EXAMPLES III (A)

Type I. Ex. 1. Find the ratios in which the coordinate planes divide the line joining $(-2, 4, 7)$ and $(3, -5, 8)$. *(Pakistan, 1957)*

Sol. The equations of the coordinate planes are

$$x=0 \quad ...(1), \qquad y=0 \quad ...(2), \qquad z=0 \quad ...(3).$$

Let the required ratios be $\lambda_1 : 1, \lambda_2 : 1, \lambda_3 : 1$ respectively,

The points which divide the join of $(-2, 4, 7)$ and $(3, -5, 8)$ in the ratios $\lambda_1 : 1, \lambda_2 : 1, \lambda_3 : 1$ repectively are

$$P\left[\frac{3\lambda_1-2}{\lambda_1+1}, \frac{-5\lambda_1+4}{\lambda_1+1}, \frac{8\lambda_1+7}{\lambda_1+1}\right],$$

$$Q\left[\frac{3\lambda_2-2}{\lambda_2+1}, \frac{-5\lambda_2+4}{\lambda_2+1}, \frac{8\lambda_2+7}{\lambda_2+1}\right],$$

and $$R\left[\frac{3\lambda_3-2}{\lambda_3+1}, \frac{-5\lambda_3+4}{\lambda_3+1}, \frac{8\lambda_3+7}{\lambda_3+1}\right].$$

$\because$ P lies on (1), $\quad\therefore \quad \dfrac{3\lambda_1-2}{\lambda_1+1}=0, \quad$ or, $\lambda_1=\dfrac{2}{3}$.

$\because$ Q lies on (2), $\quad\therefore \quad \dfrac{-5\lambda_2+4}{\lambda_2+1}=0, \quad$ or, $\lambda_2=\dfrac{4}{5}$.

$\because$ R lies on (3), $\quad\therefore \quad \dfrac{8\lambda_3+7}{\lambda_3+1}=0, \quad$ or, $\lambda_3=-\dfrac{7}{8}$.

$\therefore$ required ratios are $2 : 3, 4 : 5, -7 : 8$.

Ex. 2. Show that the plane $ax+by+cz+d=0$ divides the join of (x_1, y_1, z_1) and (x_2, y_2, z_2) in the ratio

$$-(ax_1+by_1+cz_1+d)/(ax_2+by_2+cz_2+d). \qquad \textit{(Kashmir, 1958)}$$

Ex. 3. Find the ratios in which the coordinate planes divide the line joining the points

$(-1, 2, 3)$ and $(3, -4, 5)$. *(Punjab, 1960)*

[**Ans.** $1 : 3;\ 1 : 2;\ -3 : 5$]

Type II. Ex. 1. A variable plane is at a constant distance p from the origin and meets the axes in A, B, C. Through A, B, C, planes are drawn parallel to the coordinate planes. Show that the locus of their point of intersection is given by $x^{-2}+y^{-2}+z^{-2}=p^{-2}$.

[*Karnatak, 1960*; *Vikram Engg., 1959*; *Delhi Hons., 1963*; *Lucknow (Pass), 1964*]

Sol. Let $$lx+my+nz=p \qquad ...(1)$$
be the equation of the plane, where l, m, n are the actual direction cosines of the normal to the plane.

This plane meets x-axis $(y=0=z)$ at A $\left(\frac{p}{l}, 0, 0\right)$; meets y-axis $(x=0=z)$ at B $\left(0, \frac{p}{m}, 0\right)$ and meets z-axis $(x=0=y)$ at C $\left(0, 0, \frac{p}{n}\right)$.

The planes through A, B, C parallel to the coordinate planes $x=0$, $y=0$, $z=0$ respectively are given by the equations

$$x=\frac{p}{l},\ y=\frac{p}{m},\ z=\frac{p}{n}.$$

Let these planes meet in the point (λ, μ, ν).

$$\therefore \qquad \lambda=\frac{p}{l},\ \mu=\frac{p}{m} \text{ and } \nu=\frac{p}{n},$$

or

$$l=\frac{p}{\lambda},\ m=\frac{p}{\mu},\ n=\frac{p}{\nu}.$$

But $\qquad l^2+m^2+n^2=1.$

$$\therefore \qquad p^2(\lambda^{-2}+\mu^{-2}+\nu^{-2})=1.$$

$\therefore$ locus of (λ, μ, ν) is

$$x^{-2}+y^{-2}+z^{-2}=p^{-2}.$$

Ex. 2. A variable plane which remains at a constant distance $3p$ from the origin, cuts the coordinate axes in A, B, C. Show that the locus of the centroid of the triangle ABC is $x^{-2}+y^{-2}+z^{-2}=p^{-2}$.

(*Karnatak Engg., 1961* ; *Punjab B.Sc., 1958 S.* ; *Lucknow (Pass), 1961* ; *Patna, 1963* ; *Punjab T.D.C., 1964 S*)

Ex. 3. A variable plane is at a constant distance p from the origin and meets the axes in A, B, C.

Show that the locus of the centroid of the tetrahedron OABC is

$$x^{-2}+y^{-2}+z^{-2}=16\,p^{-2}.$$

Type III. Ex. 1. A point P moves on the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ which is fixed, and the plane through P perpendicular to OP meets the axes in A, B, C. If the planes through A, B, C parallel to the coordinate planes meet in a point Q, show that the locus of Q is

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{ax}+\frac{1}{by}+\frac{1}{cz}.$$

(*Allahabad, 1962*)

Sol. The given fixed plane is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \qquad \ldots(1)$$

Let the point P be (α, β, γ).

$\because$ P lies on (1),

$$\therefore \qquad \frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}=1 \qquad \ldots(2)$$

Any plane through P is

$$A(x-\alpha)+B(y-\beta)+C(z-\gamma)=0 \qquad ...(3)$$

If it is perpendicular to OP, then the normal to the plane, whose direction ratios are A, B, C, is parallel to OP, whose direction ratios are α, β, γ.

$$\therefore \qquad \frac{A}{\alpha}=\frac{B}{\beta}=\frac{C}{\gamma}.$$

$\therefore$ (3) becomes $\alpha(x-\alpha)+\beta(y-\beta)+\gamma(z-\gamma)=0$,

or,
$$\alpha x+\beta y+\gamma z=\alpha^2+\beta^2+\gamma^2.$$

It meets the axes of coordinates in points A, B, C whose coordinates are respectively

$$\left(\frac{\alpha^2+\beta^2+\gamma^2}{\alpha}, 0, 0\right), \left(0, \frac{\alpha^2+\beta^2+\gamma^2}{\beta}, 0\right), \left(0, 0, \frac{\alpha^2+\beta^2+\gamma^2}{\gamma}\right).$$

Planes through A, B, C parallel to YOZ, ZOX and XOY planes are respectively

$$x=\frac{\alpha^2+\beta^2+\gamma^2}{\alpha}, \qquad y=\frac{\alpha^2+\beta^2+\gamma^2}{\beta},$$
$$z=\frac{\alpha^2+\beta^2+\gamma^2}{\gamma}$$

Let these planes meet in Q (λ, μ, ν).

$$\therefore \qquad \lambda=\frac{\alpha^2+\beta^2+\gamma^2}{\alpha}, \quad \mu=\frac{\alpha^2+\beta^2+\gamma^2}{\beta}, \quad \nu=\frac{\alpha^2+\beta^2+\gamma^2}{\gamma}.$$

Now,
$$\frac{1}{\lambda^2}+\frac{1}{\mu^2}+\frac{1}{\nu^2}=\frac{\alpha^2+\beta^2+\gamma^2}{(\alpha^2+\beta^2+\gamma^2)^2}=\frac{1}{\alpha^2+\beta^2+\gamma^2}$$

and
$$\frac{1}{a\lambda}+\frac{1}{b\mu}+\frac{1}{c\nu}=\frac{\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}}{\alpha^2+\beta^2+\gamma^2}$$
$$=\frac{1}{\alpha^2+\beta^2+\gamma^2}, \qquad \text{using (2).}$$

$$\therefore \qquad \frac{1}{\lambda^2}+\frac{1}{\mu^2}+\frac{1}{\nu^2}=\frac{1}{a\lambda}+\frac{1}{b\mu}+\frac{1}{c\nu}.$$

$\therefore$ locus of (λ, μ, ν) is

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{ax}+\frac{1}{by}+\frac{1}{cz}.$$

Ex 2. A plane triangle, sides a, b, c is placed so that the middle points of the sides are on the axes. Show that if α, β, γ are the intercepts on the axes, $\alpha^2=\frac{1}{8}(b^2+c^2-a^2)$, $\beta^2=\frac{1}{8}(a^2+c^2-b^2)$, $\gamma^2=\frac{1}{8}(a^2+b^2-c^2)$ and that the coordinates of the vertices are $(-\alpha, \beta, \gamma)$, $(\alpha, -\beta, \gamma)$ and $(\alpha, \beta, -\gamma)$.

Type IV. Ex. 1. Find the equation of the plane through the three points (0, 1, 1), (1, 1, 2), (–1, 2, –2). [*Punjab (Pakistan), 1957*]

Sol. Here
$$x_1=0, \quad x_2=1, \quad x_3=-1.$$
$$y_1=1, \quad y_2=1, \quad y_3=2.$$
$$z_1=1, \quad z_2=2, \quad z_3=-2.$$

Using Art. 3·8, we have

$$\begin{vmatrix} x & y & z & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ -1 & 2 & -2 & 1 \end{vmatrix} = 0,$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{vmatrix} + 1 \begin{vmatrix} y & z & 1 \\ 1 & 1 & 1 \\ 2 & -2 & 1 \end{vmatrix} + 1 \begin{vmatrix} y & z & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = 0,$$

or, $$x[(2+2)-1(1-2)+1(-2-4)]+[y(1+2)-z(1-2)+1(-2-2)]$$
$$+[y(1-2)-z(1-1)+1(2-1)]=0,$$

or, $$-x+3y+z-4-y+1=0, \quad \text{or,} \quad x-2y-z+3=0.$$

Ex. 2. Find the equation of the plane through the following points : *(Jodhpur Engg., 1963)*

(*i*) $(1, 1, 0)$, $(-2, 2, -1)$ and $(1, 2, 1)$.

Also find the intercepts the plane makes on the axes. *(Jodhpur Engg., 1963)*

(*ii*) $(8, -2, 2)$, $(2, 1, -4)$ and $(2, 4, -6)$.

[**Ans.** (*i*) $2x+3y-3z-5=0$, $\frac{5}{2}$, $\frac{5}{3}$, $-\frac{5}{3}$; (*ii*) $2x-2y-3z=14$.]

Type V. Ex. 1. Prove that the four points $(0, -1, 0)$, $(2, 1, -1)$, $(1, 1, 1)$ and $(3, 3, 0)$ are coplanar.
(Delhi Hons., 1950 ; Nagpur, 1955S; Karnatak Engg., 1961; Punjab B.Sc., 1954 S.)

Sol. Let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be the unit vectors along the positive axes of x, y and z of a right-handed system.

Let the four points be A, $-\mathbf{j}$; B, $2\mathbf{i}+\mathbf{j}-\mathbf{k}$; C, $\mathbf{i}+\mathbf{j}+\mathbf{k}$; D, $3\mathbf{i}+3\mathbf{j}$ respectively.

From (A) of Art. 3·8, the equation of the plane ABC is

$$\mathbf{r}=(1-s-t)(-\mathbf{j})+s(2\mathbf{i}+\mathbf{j}-\mathbf{k})+t(\mathbf{i}+\mathbf{j}+\mathbf{k}) \qquad \ldots(1)$$

If D lies on (1), then

$$3\mathbf{i}+3\mathbf{j}=(2s+t)\mathbf{i}+(2s+2t-1)\,\mathbf{j}+(t-s)\mathbf{k}.$$

Equating coefficients of like vectors, we have

$$2s+t=3, \qquad 2s+2t-1=3, \qquad t-s=0.$$

Solving first two equations,

$$s=1, \ t=1.$$

These values satisfy the third equation.

Hence A, B, C, D are coplanar.

Aliter. The plane through the points (0, −1, 0), (2, 1, −1) and (1, 1, 1) is

$$\begin{vmatrix} x & y & z & 1 \\ 0 & -1 & 0 & 1 \\ 2 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0,$$

or, $$x\begin{vmatrix} -1 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} + 2\begin{vmatrix} y & z & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} - 1\begin{vmatrix} y & z & 1 \\ -1 & 0 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 0,$$

or, $$x[-1(-1-1)+1(1+1)]$$
$$+2[y(0-1)-z\,(-1-1)+(-1-0)]-1[y(0+1)-z(-1-1)+1(1-0)]=0$$

or, $$4x-2y+4z-2-y-2z-1=0,$$

or, $$4x-3y+2z-3=0.$$

The point (3, 3, 0) lies on it. Hence the given four points are coplanar.

Ex. 2. Prove that the following points are coplanar :

(*i*) (0, −1, −1), (4, 5, 1), (3, 9, 4), (−4, 4, 4).

(*ii*) (1, 1, 0), (−2, 2, −1), (1, 2, 1), (1, 0, −1).

(*Agra, 1960* ; *Bihar, 1960* ; *Punjab B.Sc , 1965*)

Type VI. Ex. 1. Find the equation of the plane that passes through (2, −3, 1) and is perpendicular to the line joining the points (3, 4, −1), (2, −1, 5). (*Punjab, 1964* ; *Ranchi, 1963*)

Sol. The direction ratios of the join of the points (3, 4, −1) and (2, −1, 5) are 3−2, 4−(−1), −1−5, or 1, 5, −6.

Now any plane perpendicular to this line will have 1, 5, −6 as the coefficients of x, y, z respectively in its equation.

Let the plane be $x+5y-6z = k$. ...(1)

$\because$ it passes through (2, −3, 1), $\therefore$ $k=-19$.

$\therefore$ (1) becomes $x+5y-6z+19=0$,

Ex. 2. O is the origin and A is the point (a, b, c). Find the direction cosines of OA and deduce the equation of the plane through A at right angles to OA. (*Patna, 1960*)

[**Ans.** $ax+by+cz=a^2+b^2+c^2$.]

Ex. 3. If the axes are rectangular, and P is the point (2, 3, −1), find the equation of the plane through P at right angles to OP.

(*A.M.I.E., Nov. 1956* ; *Patna, 1962 S.*; *Sagar B.Sc., 1962*)

[**Ans.** $2x+3y-z=14$.]

SECTION II

ANGLE FORMULA

3·10. Angle between two planes : Def.

The angle between two planes π and α is the angle which the positive direction of a normal to the plane π makes with the positive direction of a normal to the plane α.

3·11. To find the angle between the two planes $A_1x+B_1y+C_1z+D_1=0$ and $A_2x+B_2y+C_2z+D_2=0$.

The given planes are $A_1x+B_1y+C_1z+D_1=0$. ...(1)

and $A_2x+B_2y+C_2z+D_2=0$. ...(2)

Let θ be the angle between (1) and (2).

The angle between (1) and (2)

$=$the angle between their normals.

The direction ratios of the normals of (1) and (2) are respectively

$$A_1, B_1, C_1 \text{ and } A_2, B_2, C_2.$$

$$\therefore \quad \cos\theta=\frac{A_1A_2+B_1B_2+C_1C_2}{\sqrt{A_1^2+B_1^2+C_1^2}\ \sqrt{A_2^2+B_2^2+C_2^2}}.$$

Hence $\quad \theta=\cos^{-1}\left[\dfrac{A_1A_2+B_1B_2+C_1C_2}{\sqrt{A_1^2+B_1^2+C_1^2}\ \sqrt{A_2^2+B_2^2+C_2^2}}\right].$

Note. 1. Complete angle formula.

The complete angle formula is given by

$$\cos\theta=\pm\left(\frac{A_1A_2+B_1B_2+C_1C_2}{\sqrt{A_1^2+B_1^2+C_1^2}\ \sqrt{A_2^2+B_2^2+C_2^2}}\right)$$

Note 2. Angle formula in terms of tan θ.

$$\cos\theta=[A_1A_2+B_1B_2+C_1C_2]/\sqrt{A_1^2+B_1^2+C_1^2}\sqrt{A_2^2+B_2^2+C_2^2}$$

Also, $\sin\theta=\dfrac{\sqrt{(B_1C_2-B_2C_1)^2+(C_1A_2-C_2A_1)^2+(A_1B_2-A_2B_1)^2}}{\sqrt{A_1^2+B_1^2+C_1^2}\ \sqrt{A_2^2+B_2^2+C_2^2}}.$

$$\therefore \quad \tan\theta=[\sqrt{(B_1C_2-B_2C_1)^2+(C_1A_2-C_2A_1)^2+(A_1B_2-A_2B_1)^2}]/(A_1A_2+B_1B_2+C_1C_2)$$

3·12. Condition of parallelism and perpendicularity. Find the condition that the planes $A_1x+B_1y+C_1z+D_1=0$ and $A_2x+B_2y+C_2z+D_2=0$ are (i) parallel and (ii) perpendicular.

Let θ be the angle between the planes.

(*i*) If the planes are parallel, $\theta=0$.

$\therefore\quad \tan\theta = \tan 0 = 0, \qquad$ or,

$$[\sqrt{\Sigma(B_1C_2-B_2C_1)^2}/(A_1A_2+B_1B_2+C_1C_2)]=0,$$

or, $(B_1C_2-B_2C_1)^2+(C_1A_2-C_2A_1)^2+(A_1B_2-A_2B_1)^2=0$

$\therefore\quad B_1C_2-B_2C_1=0,\ C_1A_2-C_2A_1=0,\ A_1B_2-A_2B_1=0$

$\therefore\quad \dfrac{A_1}{A_2}=\dfrac{B_1}{B_2}=\dfrac{C_1}{C_2}$, which is the required condition.

(*ii*) If the planes are perpendicular, $\theta=90°$,

$$\therefore\qquad \cos\theta=\cos 90°=0,$$

or
$$A_1A_2+B_1B_2+C_1C_2=0,$$
which is the required condition.

Note 1. *The converse of the above is also true.*

Note 2. *The equation of any plane parallel to the plane $Ax+By+Cz+D=0$ is* $\mathbf{Ax+By+Cz+\lambda=0}$, *where λ is determined from the additional condition provided in the problem.*

EXAMPLES III (B)

Type I. Ex. 1. Find the angle between the planes $2x-y+z=6$, $x+y+2z=3$. *(Delhi Hons., 1945)*

Sol. Here $A_1=2,\ B_1=-1,\ C_1=1,\ D_1=-6$;
$A_2=1,\ B_2=1,\ C_2=2,\ D_2=-3.$

Let θ be the required angle.

$$\therefore\ \cos\theta=\frac{2.1+(-1).1+1(2)}{\sqrt{2^2+(-1)^2+1^2}\ \sqrt{1^2+1^2+(2)^2}}=\frac{3}{6}=\frac{1}{2}.$$

$\therefore\qquad \theta=60°.$

Ex. 2. Find the angle between the planes :

(*i*) $2x+2y+z=5,\ 3x-2y-z=6.$

(*ii*) $x-y+z+1=0,\ y-2x-6z=5.$

$$\left[\textbf{Ans.}\ \cos^{-1}\left(\frac{1}{3\sqrt{14}}\right);\ \cos^{-1}\left(\frac{9}{\sqrt{123}}\right).\right]$$

Ex. 3. Prove that the planes

(*i*) $3x-2y+z+17=0$

and $4x+3y-6z+25=0$ are at right angles. *(Raj. Engi., 1961)*

(*ii*) $x+2y+2z=0,\ 2x+y-2z=0$ are orthogonal. Also find another plane through the origin which will be perpendicular to each of the planes. *(Calcutta, 1961)*

[**Ans.** $2x-2y+z=0$.]

Type II. Ex. (i) Find the equation of the plane through $(0, 1, -2)$ parallel to the plane $2x-3y+4z=0$.

(ii) Find the equation of the plane through $(-1, 1, 1)$ and $(1, -1, 1)$ perpendicular to the plane $x+2y+2z=5$. *(Bihar, 1960S ; Bhagalpur, 1962S)*

Sol. (*i*) Equation of any plane parallel to the given plane is

$$2x-3y+4z+\lambda=0 \qquad ...(1)$$

$\because$ (1) passes through (0, 1, −2), $\quad\therefore\quad \lambda=11$.

$\therefore$ (1) becomes $2x-3y+4z+11=0$.

(*ii*) Let the required plane be $Ax+By+Cz+D=0 \qquad ...(1)$

$\because$ (1) is perpendicular to the plane $x+2y+2z=5$,

$$\therefore \qquad A+2B+2C=0 \qquad ...(2)$$

$\because$ (1) passes through (−1, 1, 1) and (1, −1, 1),

$$\therefore \qquad -A+B+C+D=0 \qquad ...(3)$$

and

$$A-B+C+D=0 \qquad ...(4)$$

Subtracting (4) from (3), we have

$$-2A+2B=0, \quad \text{or} \quad A=B \qquad ...(5)$$

From (2), we have $\qquad 3B=-2C,$

or

$$B=-\frac{2C}{3}.$$

$$\therefore \qquad A=-\frac{2C}{3}.$$

From (3),

$$D=-\frac{2C}{3}+\frac{2C}{3}-C=-C.$$

$\therefore$ (1) becomes $-\dfrac{2C}{3}x-\dfrac{2C}{3}y+Cz-C=0,$

or,

$$2x+2y-3z+3=0.$$

Ex. 2. Find the equation of the plane through the

(*i*) point (2, −3, 4) and parallel to the plane $2x-5y-7z=6$. (*Bihar, 1961*)

(*ii*) point (1, 2, 3) and parallel to the plane $3x+4y-5z=0$. (*Patna, 1962*)

(*iii*) points (2, 2, 1), (9, 3, 6) and perpendicular to the plane

$$2x+6y+6z=9.$$

(*Patna, 1961 S*)

[**Ans.** (*i*) $2x-5y-7z+9=0$;
(*ii*) $3x+4y-5z+4=0$;
(*iii*) $3x+4y-5z-9=0$.]

Ex. 3. Find the equation of the plane through (*i*) (2, −3, 1) normal to the line joining the points (3, 4, −1) and (3, −1, 5). (*Calcutta, 1963*)

[**Ans.** $5y-6z+21=0$.]

(*ii*) the point (2, 5, −8) and perpendicular to each of the planes
$2x-3y+4z+1=0$, $4x+y-2z+6=0$. (*Calcutta, 1962*)

[**Ans.** $x+10y+7z+4=0$.]

Ex. 4. Find the equation to the plane through the point (−1, 3, 2) and perpendicular to the planes $x+2y+2z=5$ and $3x+3y+2z=8$.

(*Lucknow (Pass), 1961*)

[**Ans.** $2x-4y+3z+8=0$.]

SECTION III

POSITION OF POINTS WITH RESPECT TO A GIVEN PLANE. PERPENDICULAR DISTANCE FORMULA

3·13. To show that the points (x_1, y_1, z_1) and (x_2, y_2, z_2) lie on the same or opposite sides of the plane $Ax+By+Cz+D=0$ according as the expressions $Ax_1+By_1+Cz_1+D$ and $Ax_2+By_2+Cz_2+D$ are of the same or opposite signs.

Proof. Let the given plane π be $Ax+By+Cz+D=0$ ...(1)

Let P and Q be the given points (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively.

Let the plane π divide PQ in the ratio $\lambda : 1$.

The point R which divides PQ in the ratio $\lambda : 1$ is

$$\left(\frac{\lambda x_2+x_1}{\lambda+1}, \frac{\lambda y_2+y_1}{\lambda+1}, \frac{\lambda z_2+z_1}{\lambda+1}\right).$$

$\because$ it lies on the plane (1),

$$\therefore \quad A\left(\frac{\lambda x_2+x_1}{\lambda+1}\right)+B\left(\frac{\lambda y_2+y_1}{\lambda+1}\right)+C\left(\frac{yz_2+z_1}{\lambda+1}\right)+D=0,$$

or, $\quad A(\lambda x_2+x_1)+B(\lambda y_2+y_1)+C(\lambda z_2+z_1)+D(\lambda+1)=0,$

or, $\quad \lambda(Ax_2+By_2+Cz_2+D)+(Ax_1+By_1+Cz_1+D)=0,$

or $\lambda=-(Ax_1+By_1+Cz_1+D)/(Ax_2+By_2+Cz_2+D)$ ...(2)

Now (*i*) If $Ax_1+By_1+Cz_1+D$, $Ax_2+By_2+Cz_2+D$ are of the same sign, then from (2), λ is negative.

$\therefore$ PQ is divided externally by the plane,

i.e., P and Q lie on the same side of the plane (1).

(*ii*) If $Ax_1+By_1+Cz_1+D$ and $Ax_2+By_2+Cz_2+D$ are of opposite signs, then from (2), λ is positive.

$\therefore$ plane (1) divides PQ internally,

i.e., P and Q lie on opposite sides of the plane (1).

This proves the proposition.

Cor. In case D is positive, the expression $Ax_1+By_1+Cz_1+D$ is positive if (x_1, y_1, z_1) and the origin lie on the same side of the plane

$$Ax+By+Cz+D=0,$$

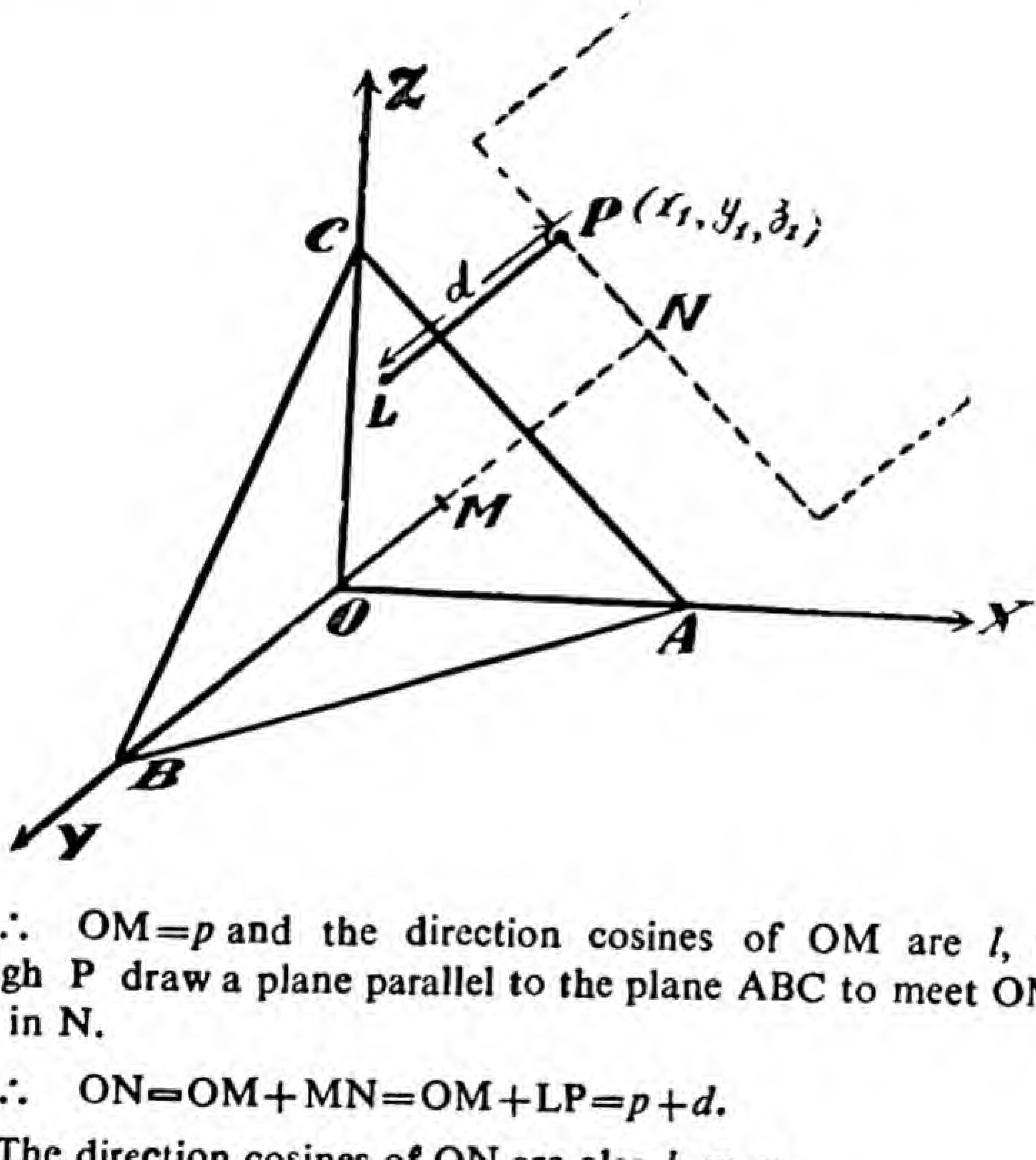
and negative, if they lie on opposite sides.

3·14. To find the length of the perpendicular from the given point (x_1, y_1, z_1) upon the plane $lx+my+nz=p$.

Let P be the point (x_1, y_1, z_1) and ABC, the plane

$$lx+my+nz=p \qquad \text{...(1)}$$

Draw PL perpendicular on the plane ABC. Let PL be d. Let OM be perpendicular from O on the plane ABC.



$\therefore$ OM$=p$ and the direction cosines of OM are l, m, n. Through P draw a plane parallel to the plane ABC to meet OM produced in N.

$\therefore$ ON$=$OM$+$MN$=$OM$+$LP$=p+d$.

The direction cosines of ON are also l, m, n.

$\therefore$ equation of this plane parallel to (1) is

$$lx+my+nz=p+d.$$

16470.

$\because$ it passes through P,

$\therefore$ $lx_1+my_1+nz_1=p+d$.

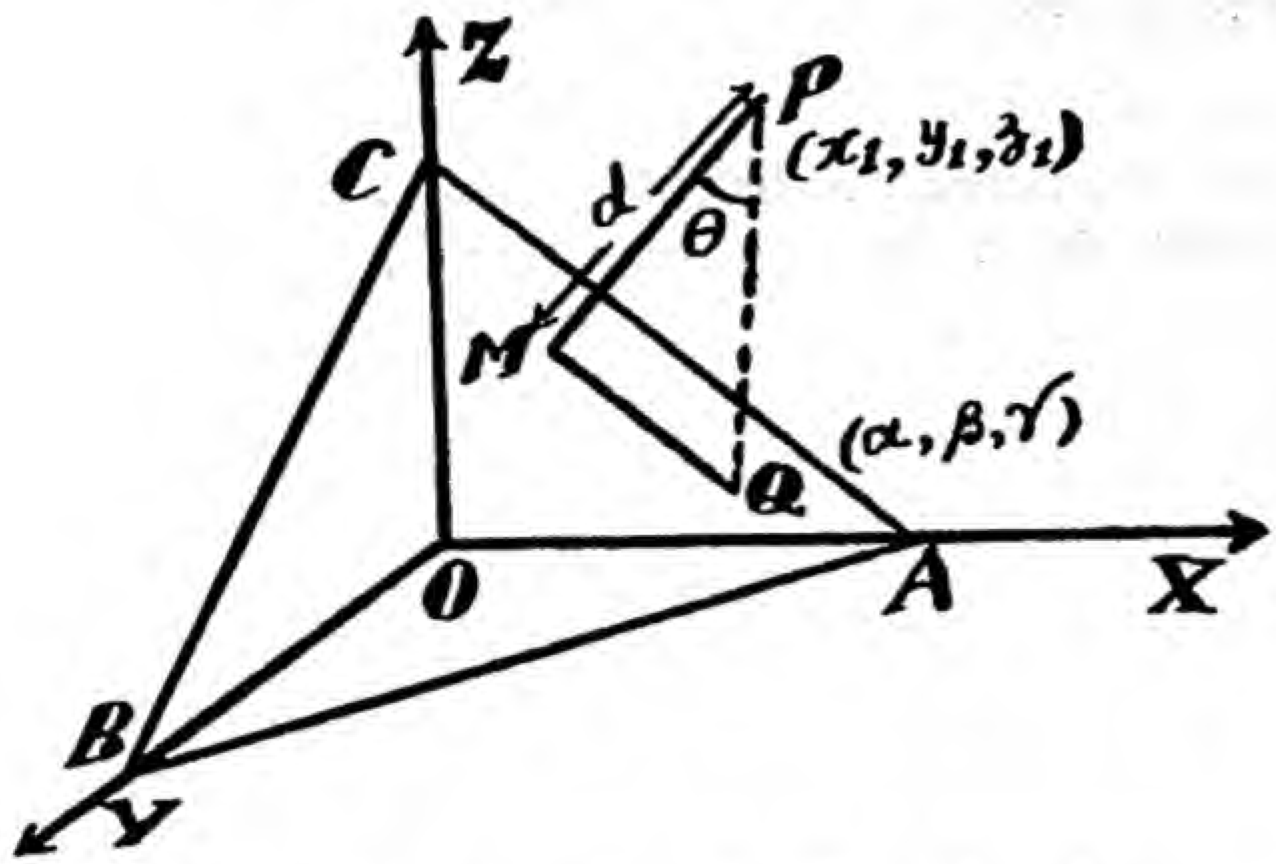
Hence $$\mathbf{d=lx_1+my_1+nz_1-p.}$$

Note. *In the above investigation we have taken O and P on opposite sides of the plane. If they lie on the same side of the plane, then we shall have*

$$d=-(lx_1+my_1+nz_1-p).$$

3·15. To find the length of the perpendicular from the given point (x_1, y_1, z_1) upon the plane

$$Ax+By+Cz+D=0.$$



Let P be the point (x_1, y_1, z_1) and ABC, the plane

$$Ax+By+Cz+D=0 \qquad \ldots(1)$$

Draw PM perpendicular to the plane ABC.

Let PM be d.

Let us take $Q(\alpha, \beta, \gamma)$ any point on the plane ABC.

$$\therefore \qquad A\alpha+B\beta+C\gamma+D=0 \qquad \ldots(2)$$

Join PQ and MQ.

Let θ be the angle MPQ.

Now, angle $\angle PMQ=90°$.

$$\therefore \qquad PM=PQ\cos\theta \qquad \ldots(3)$$

But PM is normal to the plane ABC.

$\therefore$ its direction ratios are A, B, C,

or, its direction cosines are

$$\frac{A}{\sqrt{A^2+B^2+C^2}}, \quad \frac{B}{\sqrt{A^2+B^2+C^2}}, \quad \frac{C}{\sqrt{A^2+B^2+C^2}}.$$

The direction ratios of PQ are $x_1-\alpha$, $y_1-\beta$, $z_1-\gamma$.

$\therefore$ direction cosines of PQ are

$$\frac{x_1-\alpha}{\sqrt{\Sigma(x_1-\alpha)^2}}, \quad \frac{y_1-\beta}{\sqrt{\Sigma(x_1-\alpha)^2}}, \quad \frac{z_1-\gamma}{\sqrt{\Sigma(x_1-\alpha)^2}}$$

$$\therefore \quad \cos\theta = \frac{A(x_1-\alpha)+B(y_1-\beta)+C(z_1-\gamma)}{\sqrt{A^2+B^2+C^2}\ \sqrt{\Sigma(x_1-\alpha)^2}}$$

Also, $\quad PQ=\sqrt{\Sigma(x_1-\alpha)^2}$.

$\therefore$ (3) gives

$$d=\frac{A(x_1-\alpha)+B(y_1-\beta)+C(z_1-\gamma)}{\sqrt{A^2+B^2+C^2}\sqrt{\Sigma(x_1-\alpha)^2}}\cdot\sqrt{\Sigma(x_1-\alpha)^2}$$

$$=\frac{Ax_1+By_1+Cz_1-(A\alpha+B\beta+C\gamma)}{\sqrt{A^2+B^2+C^2}}.$$

Hence $\quad \mathbf{d}=\dfrac{\mathbf{Ax_1+By_1+Cz_1+D}}{\sqrt{\mathbf{A^2+B^2+C^2}}}$, using (2).

Note. Complete perpendicular distance formula.

The complete perpendicular distance formula is

$$d=\pm(Ax_1+By_1+Cz_1+D)/\sqrt{A^2+B^2+C^2}.$$

EXAMPLES III (C)

Type I. Ex. 1. Show that the two points (2, 3, −5) and (3, 4, 7) lie on opposite sides of the plane meeting the axes in A, B, C such that the centroid of the △ABC is the point (1, 2, 4). *(Raj. Engg., 1963S.)*

Sol. Let the plane be

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \qquad ...(1)$$

meeting the axes in A, B, C respectively.

$\therefore$ co-ordinates of A, B, C are respectively

$(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$.

By the problem,

$$\frac{a+0+0}{3}=1,\quad \frac{0+b+0}{3}=2,$$

$$\frac{0+0+c}{3}=4, \text{ or } a=3,\ b=6,\ c=12.$$

$\therefore$ (1) becomes $\quad \dfrac{x}{3}+\dfrac{y}{6}+\dfrac{z}{12}-1=0 \qquad ...(2)$

Substituting the points (2, 3, −5) and (3, 4, 7) in the L.H.S. of (2), we have

$$\frac{2}{3}+\frac{3}{6}-\frac{5}{12}-1 \quad\text{and}\quad \frac{3}{3}+\frac{4}{6}+\frac{7}{12}-1$$

or $\qquad -\frac{1}{4}$ and $\frac{9}{4}$.

$\because$ the results are of the opposite signs,

$\therefore$ the points (2, 3, −5) and (3, 4, 7) lie on the opposite sides of the plane.

Ex. 2. Show that the points (1, 1, 1) and (— 3, 0, 1) lie on opposite sides of the plane $3x+4y-12z+13=0$.

Ex. 3. Are the points (2, 1, 1) and (2, 5, —1) lie on the same or opposite sides of the plane $x-2y-3z+4=0$? [**Ans.** On opposite sides]

Type II. Ex. 1. Two systems of rectangular axes have the same origin. If a plane cuts them at distances a, b, c and a′, b′, c′ respectively from the origin, prove that

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{a'^2}+\frac{1}{b'^2}+\frac{1}{c'^2}.$$

(*Agra, 1953 ; Delhi Hons., 1956 ; A.M.I.E. May, 1960 ; Vikram 1962 ; Aligarh, 1961*)

Sol. Let OX, OY, OZ and OX_1, OY_1, OZ_1 be the two systems of rectangular axes with a common origin O.

The equation of the plane with respect to the axes OX, OY, OZ is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \qquad ...(1)$$

and the equation of the plane with respect to the axes OX_1, OY_1, OZ_1 is

$$\frac{x}{a'}+\frac{y}{b'}+\frac{z}{c'}=1 \qquad ...(2)$$

We know that from a point external to a plane one and only one perpendicular can be drawn.

∴ the lengths of the perpendiculars from the origin on (1) and (2) representing the same plane must be equal.

$$\therefore \quad \frac{1}{\sqrt{a^{-2}+b^{-2}+c^{-2}}}=\frac{1}{\sqrt{a'^{-2}+b'^{-2}+c'^{-2}}}, \qquad \text{(Art. 3·15)}$$

or

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{a'^2}+\frac{1}{b'^2}+\frac{1}{c'^2}.$$

Ex. 2. The sum of the squares of the distances of a point from the planes

$$x+y+z=0,$$
$$x-2y+z=0$$

is equal to the square of its distance from the plane $x=z$. Prove that the equation of the locus of the point is

$$y^2+2zx=0. \qquad \textit{(Delhi Hons., 1955 ; Patna, 1959 S.)}$$

Ex. 3. Find the locus of a point the sum of the squares of whose distances from the planes $x+y+z=0$,

$$x-z=0,$$
$$x-2y+z=0, \text{ is } 9. \qquad [\textbf{Ans.}\ x^2+y^2+z^2=9.]$$

(*Karnatak Engg., 1961*)

Type III. Ex. 1. Find the distance between the parallel planes

$$\mathbf{2x-2y+z+1=0}$$

and

$$\mathbf{4x-4y+2z+3=0.}$$

Sol. Let (x_1, y_1, z_1) be **any** point on the plane

$$2x-2y+z+1=0 \qquad ...(1)$$

$$\therefore \quad 2x_1-2y_1+z_1+1=0 \qquad ...(2)$$

Now, length of the perpendicular from (x_1, y_1, z_1) on the plane

$$4x-4y+2z+3=0 \text{ is}$$

$$\frac{4x_1-4y_1+2z_1+3}{\sqrt{16+16+4}},$$

or,
$$\frac{2(2x_1-2y_1+z_1)+3}{6}.$$

$\therefore$ the perpendicular distance between the given planes

$$=\frac{2(2x_1-2y_1+z_1)+3}{6}=\frac{2(-1)+3}{6}, \text{ using (2).}$$

$$=\frac{1}{6}.$$

Ex. 2. Find the distance between the parallel planes :

(*i*) $2x-2y+z+6=0$, $4x-4y+2z+7=0$.

(*ii*) $x+2y+2z+3=0$, $x+2y+2z+7=0$.

(*iii*) $x-2y+2z-8=0$, $x-2y+2z+19=0$.

[**Ans.** $\frac{5}{6}$; $\frac{4}{3}$; 9.]

SECTION IV

PLANES BISECTING ANGLES BETWEEN TWO GIVEN PLANES

3·16. To find the equations of the planes bisecting the angles between the given planes

$$\mathbf{Ax+By+Cz+D=o}$$

and
$$\mathbf{A'x+B'y+C'z+D'=o.}$$

Let $P(\lambda, \mu, \nu)$ be **any** point on either of the bisecting planes.

Draw PM and PN perpendicular from P on the given planes

$$Ax+By+Cz+D=0 \qquad ...(1)$$

and
$$A'x+B'y+C'z+D'=0 \qquad ...(2)$$

$$\therefore \qquad PM=PN,$$

or,
$$\pm\left(\frac{A\lambda+B\mu+C\nu+D}{\sqrt{A^2+B^2+C^2}}\right)=\pm\left(\frac{A'\lambda+B'\mu+C'\nu+D'}{\sqrt{A'^2+B'^2+C'^2}}\right)$$

or,
$$\left(\frac{A\lambda+B\mu+C\nu+D}{\sqrt{A^2+B^2+C^2}}\right)=\pm\left(\frac{A'\lambda+B'\mu+C'\nu+D'}{\sqrt{A'^2+B'^2+C'^2}}\right).$$

$\therefore$ locus of (λ, μ, ν) is

$$\left(\frac{\mathbf{Ax+By+Cz+D}}{\sqrt{\mathbf{A^2+B^2+C^2}}}\right)=\pm\left(\frac{\mathbf{A'x+B'y+C'z+D'}}{\sqrt{\mathbf{A'^2+B'^2+C'^2}}}\right),$$

which are the required equations of the bisecting planes.

Note 1. To distinguish between the two bisecting planes. *Let P be **any** point on the plane bisecting that angle between the given planes which contains the origin.*

$\therefore$ *P and origin O lie on the same side of both the planes.*

$\therefore$ *if the equations of the planes be*

$$Ax+By+Cz+D=0$$

and $$A'x+B'y+C'z+D'=0,$$

where D and D' are both positive or both negative, then the perpendiculars from P on the planes must be both positive or both negative ;

i.e., $Ax_1+By_1+Cz_1+D$ *and* $A'x_1+B'y_1+C'z_1+D'$ *must be both positive or both negative.*

$\therefore$ *the bisecting plane of the angle between the given planes which contains the origin is*

$$\frac{Ax+By+Cz+D}{\sqrt{A^2+B^2+C^2}}=\frac{A'x+B'y+C'z+D'}{\sqrt{A'^2+B'^2+C'^2}},$$

provided D and D' are both of the same sign.

The equation of the other bisecting plane is

$$\frac{Ax+By+Cz+D}{\sqrt{A^2+B^2+C^2}}=-\frac{A'x+B'y+C'z+D'}{\sqrt{A'^2+B'^2+C'^2}}.$$

Note 2. (*i*) *If the origin lies in the acute angle between the planes, then the angle between the normals to the planes is obtuse, i.e., cos θ is negative.*

$\therefore$ $AA'+BB'+CC'$ *is negative.*

(*ii*) *If the origin lies in the obtuse angle between the given planes, then the angle between the normals to the planes is acute, i.e., cos θ is positive.*

$\therefore$ $AA'+BB'+CC'$ *is positive.*

(*The converse is also true.*)

EXAMPLES III (D)

Type I. Ex. 1. Find the bisecting plane of that angle between the planes $3x-6y+2z+5=0$ and $4x-12y+3z-3=0$ which contains the origin.

Sol. The equations of the given planes are

$$4x-12y+3z-3=0 \qquad ...(1)$$

and $$-3x+6y-2z-5=0. \qquad ...(2)$$

$\therefore$ equation of the plane bisecting the angle between (1) and (2) which contains the origin is

$$\frac{4x-12y+3z-3}{\sqrt{16+144+9}}=+\left(\frac{-3x+6y-2z-5}{\sqrt{9+36+4}}\right)$$

or, $$7(4x-12y+3z-3)=13(-3x+6y-2z-5),$$

or, $$67x-162y+47z+44=0.$$

Ex. 2. Find the equation of the bisecting plane of the angle between the following pairs of planes which contains the origin :

(*i*) $x+2y+2z=9,\ \ 4x-3y+12z+13=0.$

(*ii*) $7x+4y+4z+3=0,\ 2x+y+2z+2=0.$

[**Ans.** $25x+17y+62z-78=0$;
$x+y-2z=3.$

Type II. Ex. 1. Find the bisector of the acute angle between the planes $2x-y+2z+3=0$, $3x-2y+6z+8=0$. *(Punjab B.Sc. Hons., 1943)*

Sol. The given planes are $2x-y+2z+3=0$ ...(1)

and $3x-2y+6z+8=0$ ...(2)

Equations of the bisecting planes are

$$\frac{2x-y+2z+3}{\sqrt{4+1+4}}=\pm\left(\frac{3x-2y+6z+8}{\sqrt{9+4+36}}\right),$$

or, $14x-7y+14z+21=\pm(9x-6y+18z+24)$

Taking the positive sign, we have

$$14x-7y+14z+21=9x-6y+18z+24,$$

or $5x-y-4z-3=0$ ...(3)

Taking the negative sign, we have

$$14x-7y+14z+21=-9x+6y-18z-24,$$

or, $23x-13y+32z+45=0$ ...(4)

Consider bisecting plane (4) and the given plane.

Let θ be the angle between them.

$$\therefore \quad \cos\theta=\frac{2(23)-(1)(-13)+2(32)}{\sqrt{4+1+4}\ \sqrt{(23)^2+(-13)^2+(32)^2}}=\frac{123}{(3)\ (\sqrt{1722})}.$$

Now, $\tan\theta=\dfrac{3\sqrt{41}}{3\sqrt{1722}}=\sqrt{\dfrac{41}{1722}}=\dfrac{1}{\sqrt{42}}$,

which is numerically less than unity.

$\therefore$ the bisecting plane (4) is the bisecting plane of the acute angle.

Ex. 2. Find the bisecting plane of the obtuse angle between the planes $3x+4y-5z+1=0$ and $5x+12y-13z=0$. [**Ans.** $14x-8y+13=0$.]

Ex. 3. Find the equations of the planes bisecting the angles between the planes $x+2y+2z=19$, $4x-3y+12z+3=0$; and point out which bisects the acute angle. [**Ans.** $x+35y-10z-256=0$, $25x+17y+62z-238=0$; the latter bisects the obtuse angle.]

Type III. Ex. 1. Find the equation of the plane which bisects the obtuse angle between the planes $x+2y-2z=9$ and $4x-3y+12z+13=0$. Does the origin lie in the acute or in the obtuse angle between the planes? *(Raj., 1959)*

Sol. The given planes are

$-x-2y+2z+9=0$ ...(1)

and $4x-3y+12z+13=0$ ...(2)

Now, $AA'+BB'+CC'=(-1)(4)+(-2)(-3)+2(12)=26$,

which is positive.

$\because$ the origin lies in the obtuse angleb etween the planes.

The bisecting plane of the angle between the given planes which contains the origin is

$$\frac{-x-2y+2z+9}{\sqrt{1+4+4}}=+\left(\frac{4x-3y+12z+13}{\sqrt{16+9+144}}\right),$$

or, $$-13x-26y+26z+117=12x-9y+36z+39,$$

or, $$25x+17y+10z-78=0 \qquad \ldots(3)$$

$\because$ the origin lies in the obtuse angle,

$\therefore$ (3) gives the bisector of the obtuse angle between (1) and (2).

Ex. 2. Show that the origin lies in the acute angle between the planes

$$x+2y+2z=9,\ 4x-3y+12z+13=0.$$

Find the plane bisecting the angles between them, and point out which bisects the acute angle. (*Punjab (Pakistan), 1956 S ; Raj., 1954*)

[**Ans.** $25x+17y+62z-78=0$
(bisecting plane of the acute angle) ;
$x+35y-10z-156=0$.]

SECTION V

PAIR OF PLANES

3·17. To find the condition that the equation $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ may represent a pair of planes.

Let $$lx+my+nz=0 \qquad \ldots(1)$$

and $$l'x+m'y+n'z=0 \qquad \ldots(2)$$

be the two planes represented by the equation

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \qquad \ldots(3)$$

$$\therefore\ ax^2+by^2+cz^2+2fyz+2gzx+2hxy$$
$$\equiv(lx+my+nz)(l'x+m'y+n'z).$$

$\therefore$ comparing coefficients of like terms, we have

$$\left.\begin{aligned} ll'&=a\\ mm'&=b\\ nn'&=c\end{aligned}\right\}\ \ldots(4) \qquad \text{and} \qquad \left.\begin{aligned} mn'+m'n&=2f\\ nl'+n'l&=2g\\ lm'+l'm&=2h\end{aligned}\right\}\ \ldots(5)$$

Now, $$\begin{vmatrix} l & l' & 0\\ m & m' & 0\\ n & n' & 0\end{vmatrix}\times\begin{vmatrix} l' & l & 0\\ m' & m & 0\\ n' & n & 0\end{vmatrix}$$

$$=\begin{vmatrix} 2ll' & lm'+l'm & l'n+ln'\\ lm'+l'm & 2mm' & mn'+m'n\\ ln'+l'n & mn'+m'n & 2nn'\end{vmatrix}$$

(**Note this step.**) [See Author's 'New Text-book of Higher Algebra'.]

or, $$\begin{vmatrix} 2ll' & lm'+l'm & l'n+ln' \\ lm'+l'm & 2mm' & mn'+m'n \\ ln'+l'n & mn'+m'n & 2nn' \end{vmatrix}=0,$$

or, $$\begin{vmatrix} 2a & 2h & 2g \\ 2h & 2b & 2f \\ 2g & 2f & 2c \end{vmatrix}=0, \text{ using (4) and (5)},$$

or, $$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0$$

or, $a(bc-f^2)-h(ch-fg)+g(hf-bg)=0,$

or, $abc-af^2-ch^2+fgh+fgh-bg^2=0,$

or, $\mathbf{abc+2fgh-af^2-bg^2-ch^2=o},$

which is the required condition.

Note. *The converse of the above is also true.*

3·18. If the equation $\mathbf{ax^2+by^2+cz^2+2fyz+2gzx+2hxy=o}$ represents a pair of planes, to find the angle between them.

Let θ be the angle between the planes $lx+my+nz=0$ and $l'x+m'y+n'z=0$ represented by the equation

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0.$$

$\therefore$ as in Art. 3·17, $\left.\begin{aligned} ll'&=a \\ mm'&=b \\ nn'&=c \end{aligned}\right\} \left.\begin{aligned} mn'+m'n&=2f \\ nl'+n'l&=2g \\ lm'+l'm&=2h \end{aligned}\right\}$

$$\therefore \tan\theta=\frac{\sqrt{(mn'-m'n)^2+(nl'-n'l)^2+(lm'-l'm)^2}}{ll'+mm'+nn'}$$

$$=\frac{\sqrt{(mn'+m'n)^2-4mm'nn'+(nl'+n'l)^2-4ll'nn'+(lm'+l'm)^2-4ll'mm'}}{ll'+mm'+nn'}$$

$$=\sqrt{4(f^2-bc)+4(g^2-ca)+4(h^2-ab)}/(a+b+c)$$

$$\therefore \quad \theta=\tan^{-1}[\mathbf{2\sqrt{f^2+g^2+h^2-bc-ca-ab}/(a+b+c)}],$$

which is the required angle.

Note. *The condition that the two planes represented by the above equation are perpendicular is* $\mathbf{a+b+c=0}$.

EXAMPLES III (E)

Ex. 1. Prove that the equation $\mathbf{2x^2-6y^2-12z^2+18yz+2zx+xy=0}$. represents a pair of planes, and find the angle between them.

(*Punjab, 1957 S ; Punjab B.Sc., 1963 ; Nagpur, 1956S, 1957 ; Karnatak, 1959*)

Sol. The given equation is

$$2x^2-6y^2-12z^2+18yz+2zx+xy=0.$$

Here $a=2, b=-6, c=-12, f=9, g=1, h=\frac{1}{2}$.

$\therefore$ $abc+2fgh-af^2-bg^2-ch^2=144+9-162+6+3=0$.

$\therefore$ it represents a pair of planes. (Art. 3·17).

Let θ be the required angle.

$$\therefore \tan\theta = 2\sqrt{81+1+\tfrac{1}{4}-72+24+12} \,/\, (-16) \quad \text{(Art. 3·18)}$$
$$= \sqrt{324+4+1-288+96+48} \,/\, (-16).$$

$\therefore$ acute angle is given by $\tan\theta - \dfrac{\sqrt{185}}{16}$.

$\therefore$ $\cos\theta = \dfrac{16}{21}$, or, $\theta = \cos^{-1}\left(\dfrac{16}{21}\right)$.

Ex. 2. (*i*) Prove that the equation $2x^2-y^2+z^2+xy+3zx-2x+y-z=0$ represents a pair of planes, and find the angle between them.

[**Ans.** $\cos^{-1}\frac{1}{3}\sqrt{2}$.]

(*ii*) Show that the equation $6x^2+4y^2-10z^2+3yz+4zx-11xy=0$ represents a pair of planes. Find the equations to the planes and the angle between them.

$\left[\textbf{Ans.}\ 3x-4y+5z=0,\ 2x-y-2z=0\ ;\ \dfrac{\pi}{2}\right]$

Ex. 3. If the equation $\phi(x, y, z) \equiv ax^2+by^2+cz^2+2fyx+2gzx+2hxy=0$, represents a pair of planes, prove that the products of the distances of the two planes from (α, β, γ) is

$$\phi(\alpha, \beta, \gamma) \div \sqrt{\Sigma a^2+4\Sigma f^2-2\Sigma bc}.$$

SECTION VI

PROJECTION ON A PLANE. AREA OF TRIANGLE. VOLUME OF TETRAHEDRON.

3·19. Orthogonal projection of an area on a plane : Def.

The orthogonal projection of an area A_1 bounded by a curve ABC……on a plane α is the area A'_1 bounded by the curve A′B′C′……, where A′, B′, C′,……are the feet of the perpendiculars from A, B, C,……on the plane α.

Note. Important result of Geometry.

Area of the projection.

The projection **A′** *of a plane area A bounded by any curve on a plane α is* **A′=A cos θ**, *where θ is the angle which the plane of the area A makes with the plane α.*

3·20. Important Theorems.

(A) If A_x, A_y and A_z be the projections of an area A on the coordinate planes respectively, then to show that $A^2=A_x^2+A_y^2+A_z^2$.

Proof. Let A_x, A_y and A_z be the projections of the area A on yz, zx and xy-planes respectively

Let l, m, n be the direction cosines of the positive direction of the normal from O to the plane of area A.

Let α be the angle which the plane of the area A makes with the yz-plane.

$\therefore \quad A_x = A \cos \alpha = Al$, since α is the angle which the normal to the plane of the area A makes with the x-axis.

Similarly, $\quad A_y = A \,.\, m$ and $A_z = A \,.\, n$.

$$\therefore \quad A^2_x + A^2_y + A^2_z = A^2(l^2+m^2+n^2)$$
$$= A^2. \qquad \because \quad (l^2+m^2+n^2=1)$$

This proves the proposition.

(B) To prove that the projection of any plane area A on a plane π is equal to the sum of the projections of A_x, A_y and A_z on the plane π, where A_x, A_y, A_z are the projections of the area on the coordinate planes, *viz.*, yz, zx, and xy-planes respectively.

Proof. Let l, m, n be the direction cosines of the normal to the plane of the area A and l_1, m_1, n_1 be the direction cosines of the positive direction of the normal to the plane of projection π. Let θ be the angle between these planes.

$$\therefore \quad \cos \theta = ll_1 + mm_1 + nn_1 \qquad (1)$$

Let A′ be the projection of the area A on the plane π.

$$\therefore \quad A' = A \cos \theta \qquad \text{(Art. 3·19, note)}$$
$$= A(ll_1+mm_1+nn_1) \qquad \text{...(2), using (1).}$$

But $\quad A_x = Al$, $A_y = Am$, $A_z = An$.

$\therefore$ (2) becomes $A' = A_x l_1 + A_y m_1 + A_z n_1$
= sum of the projections of the areas A_x, A_y and A_z on tha plane π.

This proves the proposition.

3·21. Volume of the tetrahedron.

(A) To find the volume of the tetrahedron, whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), and (x_4, y_4, z_4).

Let ABCD be a tetrahedron whose vertices A, B, C and D are respectively (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4).

Let the position vectors of the vertices A, B, C, D be **a**, **b**, **c**, **d** respectively.

$\therefore\ \mathbf{a}=x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k}, \quad b=x_2\mathbf{i}+y_2\mathbf{j}+z_2\mathbf{k}, \quad c=x_3\mathbf{i}+y_3\mathbf{j}+z_3\mathbf{k}$, $d=x_4\mathbf{i}+y_4\mathbf{j}+z_4\mathbf{k}$.

Let V be the volume of the tetrahedron A,BCD.

$\therefore\quad \overrightarrow{DA}=\mathbf{a}-\mathbf{d}, \qquad \overrightarrow{DB}=\mathbf{b}-\mathbf{d}, \qquad \overrightarrow{DC}=\mathbf{c}-\mathbf{d}.$

Volume of A, BCD$=\frac{1}{3}\triangle$BCD$\times$height of A from $\triangle$BCD.

But $\triangle\ \overrightarrow{BCD}=\frac{1}{2}\overrightarrow{DB}\times\overrightarrow{DC}=\frac{1}{2}(\mathbf{b}-\mathbf{d})\times(\mathbf{c}-\mathbf{d})$,

which represents a vector perpendicular to the plane BCD.

$\therefore$ volume of tetrahedron A, BCD

$=\frac{1}{3}(\mathbf{a}-\mathbf{d})\cdot\frac{1}{2}(\mathbf{b}-\mathbf{d})\times(\mathbf{c}-\mathbf{d})$

$=\frac{1}{6}[\mathbf{a}-\mathbf{d}, \mathbf{b}-\mathbf{d}, \mathbf{c}-\mathbf{d}]$ ...(A)

$=\frac{1}{6}[(x_1-x_4)\mathbf{i}+(y_1-y_4)\mathbf{j}+(z_1-z_4)\mathbf{k}, (x_2-x_4)\mathbf{i}+(y_2-y_4)\mathbf{j}$
$(+z_2-z_4)\mathbf{k},$
$(x_3-x_4)\mathbf{i}+(y_3-y_4)\mathbf{j}+(z_3-z_4)\mathbf{k}]$

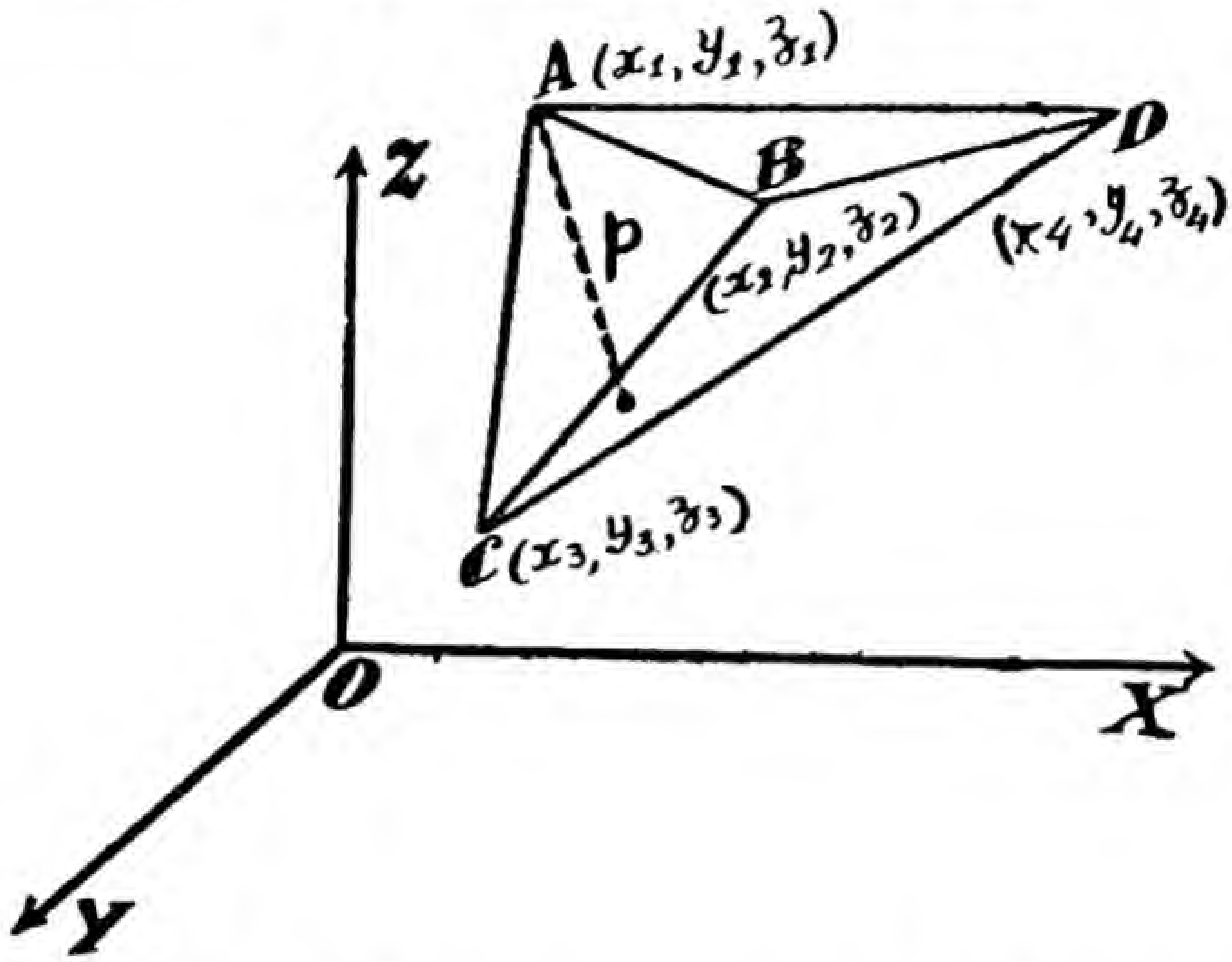
$$=\frac{1}{6}\begin{vmatrix} x_1-x_4 & y_1-y_4 & z_1-z_4 \\ x_2-x_4 & y_2-y_4 & z_2-z_4 \\ x_3-x_4 & y_3-y_4 & z_3-z_4 \end{vmatrix}=\frac{1}{6}\begin{vmatrix} 1 & x_4 & y_4 & z_4 \\ 0 & x_1-x_4 & y_1-y_4 & z_1-z_4 \\ 0 & x_2-x_4 & y_2-y_4 & z_2-z_4 \\ 0 & x_3-x_4 & y_3-y_4 & z_3-z_4 \end{vmatrix}$$

Adding first row to each of the others in succession, we have

$$V=\frac{1}{6}\begin{vmatrix} 1 & x_4 & y_4 & z_4 \\ 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \end{vmatrix}=\frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}.$$

Aliter. Let V be the volume of the tetrahedron (A, BCD) whose veatices are A (x_1, y_1, z_1), B(x_2, y_2, z_2), C(x_3, y_3, z_3) and D(x_4, y_4, z_4) respectively.

Let p be the perpendicular distance of A from the plane BCD.



(*i*) $\therefore$ $V = \frac{1}{3} \triangle BCD \times p$ ...(1) [from Pure Solid Geometry.]

[**To find p.**]

(*ii*) The equation of the plane BCD is

$$\begin{vmatrix} x & y & z & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0.$$

(*iii*) $\therefore$ p = perpendicular from A on the plane BCD.

$$= \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \div \sqrt{A^2+B^2+C^2} \quad \text{...(2)}$$

where $A = \begin{vmatrix} y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{vmatrix}$, $B = \begin{vmatrix} z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \\ z_4 & x_4 & 1 \end{vmatrix}$

and $C = \begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}$.

(*iv*) Let $\triangle$ be the area of the triangle BCD and $\triangle_x$, $\triangle_y$ and $\triangle_z$ be its projections on the coordinate planes

$x=0$, $y=0$, $z=0$ respectively.

$$\therefore \quad \triangle_x = \tfrac{1}{2} \begin{vmatrix} y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{vmatrix},$$

or,
$$\triangle_x = \frac{A}{2}, \qquad \text{or,} \qquad A = 2\triangle_x.$$

Similarly, $B = 2\triangle_y$ and $C = 2\triangle_z$

$$\therefore \quad A^2+B^2+C^2=4(\triangle_x^2+\triangle_y^2+\triangle_z^2)=4\triangle^2 \qquad ...(3)$$

using 3·20 (A), where $\triangle$ is the area of the $\triangle$ BCD.

(*v*) $\therefore$ (2) becomes

$$p = \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \div 2\triangle$$

(*vi*) $\therefore$ (1) becomes

$$V = \tfrac{1}{3}\triangle \cdot \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \cdot \frac{1}{2\triangle}.$$

Hence $\mathbf{V}=\frac{1}{6}\begin{vmatrix} \mathbf{x}_1 & \mathbf{y}_1 & \mathbf{z}_1 & 1 \\ \mathbf{x}_2 & \mathbf{y}_2 & \mathbf{z}_2 & 1 \\ \mathbf{x}_3 & \mathbf{y}_3 & \mathbf{z}_3 & 1 \\ \mathbf{x}_4 & \mathbf{y}_4 & \mathbf{z}_4 & 1 \end{vmatrix}$,

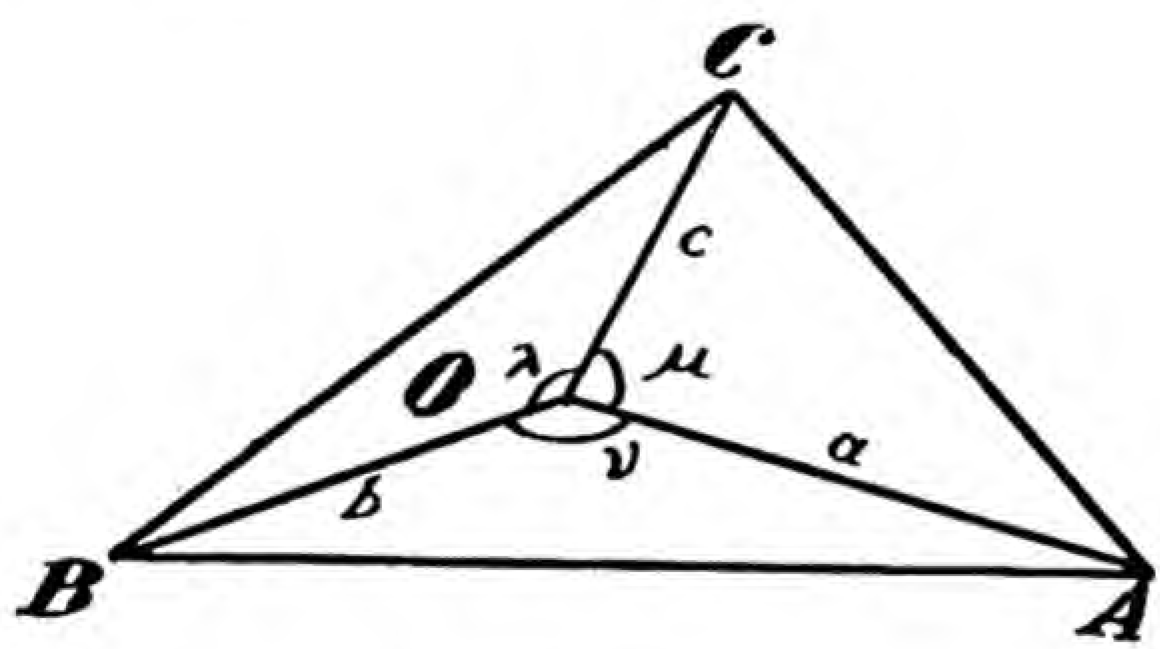
which is the required volume.

(B) To find the volume of a tetrahedron OABC, the lengths of whose edges OA, OB, OC are a, b, c and the angles BOC, COA, AOB are respectively λ, μ, ν, O being the origin.

Let O, ABC be the tetrahedron in which

$$OA=a,\ OB=b \text{ and}$$
$$OC=c.$$

Let **a**, **b**, **c** be the position vectors of A, B, C referred to O as the origin of reference.



Let (x_r, y_r, z_r), $r=1, 2, 3$ be the coordinates of the vertices A, B, C respectively.

$\therefore$ $\quad \mathbf{a}=x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k}$,

$\mathbf{b}=x_2\mathbf{i}+y_2\mathbf{j}+z_2\mathbf{k}$,

$\mathbf{c}=x_3\mathbf{i}+y_3\mathbf{j}+z_3\mathbf{k}$.

Let $\angle$BOC be λ, COA be μ and AOB be ν.

$\therefore$ $\quad a^2=\Sigma x_1^2,\ b^2=\Sigma x_2^2,\ c^2=\Sigma x_3^2$.

Also, $\quad \mathbf{a}.\mathbf{b}=\Sigma x_1x_2=\text{OA.OB} \cos \nu=ab \cos \nu$

$\mathbf{b}.\mathbf{c}=\Sigma x_2x_3=\text{OB.OC} \cos \lambda=bc \cos \lambda$

$\mathbf{c}.\mathbf{a}=\Sigma x_3x_1=\text{OC.OA} \cos \mu=ac \cos \mu$.

Let V be the volume of the tetrahedron O, ABC.

$$\therefore\quad V=\frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \qquad \text{[Art. 3·21 (A)]}$$

$$\therefore\quad V^2=\frac{1}{36}\begin{vmatrix} \Sigma x_1^2 & \Sigma x_1x_2 & \Sigma x_1x_3 \\ \Sigma x_1x_2 & \Sigma x_2^2 & \Sigma x_2x_3 \\ \Sigma x_1x_3 & \Sigma x_2x_3 & \Sigma x_3^2 \end{vmatrix}$$

$$=\frac{1}{36}\begin{vmatrix} a^2 & ab\cos\nu & ac\cos\mu \\ ab\cos\nu & b^2 & bc\cos\lambda \\ ac\cos\mu & bc\cos\lambda & c^2 \end{vmatrix}$$

$$\therefore\quad V=\frac{1}{6}\,abc\begin{vmatrix} 1 & \cos\nu & \cos\mu \\ \cos\nu & 1 & \cos\lambda \\ \cos\mu & \cos\lambda & 1 \end{vmatrix}^{\frac{1}{2}}, \text{ in magnitude.}$$

(C) Volume of the tetrahedron when equations of its four faces are given.

To find the volume of the tetrahedron the equations of whose faces are $a_rx+b_ry+c_rz+d_r=0$, $r=1, 2, 3, 4$.

The equations of the faces are

$$a_1x+b_1y+c_1z+d_1=0 \qquad \text{...(1)}$$

$$a_2x+b_2y+c_2z+d_2=0 \qquad \text{...(2)}$$

$$a_3x+b_3y+c_3z+d_3=0 \qquad \text{...(3)}$$

and

$$a_4x+b_4y+c_4z+d_4=0 \qquad \text{...(4)}$$

Solving (2), (3) and (4), we have

$$=\frac{x}{\begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \\ a_4 & c_4 & d_4 \end{vmatrix}}$$

$$=\frac{z}{\begin{vmatrix} a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \\ a_4 & b_4 & d_4 \end{vmatrix}}=\frac{-1}{\begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix}} \quad \ldots(5)$$

Let $\triangle \equiv \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$

$\therefore$ (5) becomes $\dfrac{x}{A_1}=\dfrac{-y}{-B_1}=\dfrac{z}{C_1}=\dfrac{-1}{-D_1}$,

where A_1, B_1, C_1, D_1 are the co-factors of a_1, b_1, c_1 and d_1 in $\triangle$.

$$\therefore \qquad x=\frac{A_1}{D_1},\ y=\frac{B_1}{D_1},\ z=\frac{C_1}{D_1}.$$

$\therefore$ one vertex of the tetrahedron is

$$\left(\frac{A_1}{D_1}, \frac{B_1}{D_1}, \frac{C_1}{D_1}\right).$$

Similarly, other vertices are

$$\left(\frac{A_2}{D_2}, \frac{B_2}{D_2}, \frac{C_2}{D_2}\right), \left(\frac{A_3}{D_3}, \frac{B_3}{D_3}, \frac{C_3}{D_3}\right)$$

and

$$\left(\frac{A_4}{D_4}, \frac{B_4}{D_4}, \frac{C_4}{D_4}\right).$$

$\therefore$ Volume of the tetrahedron

$$V=\frac{1}{6}\begin{vmatrix} A_1/D_1 & B_1/D_1 & C_1/D_1 & 1 \\ A_2/D_2 & B_2/D_2 & C_2/D_2 & 1 \\ A_3/D_3 & B_3/D_3 & C_3/D_3 & 1 \\ A_4/D_4 & B_4/D_4 & C_4/D_4 & 1 \end{vmatrix}$$

$$=\frac{1}{6D_1\, D_2\, D_3\, D_4}\begin{vmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{vmatrix}$$

$$=\frac{1}{6D_1 D_2 D_3 D_4}\Delta^{3*} \text{ (From Higher Algebra)}$$

Hence $\mathbf{V}=\dfrac{1}{6D_1 D_2 D_3 D_4}\Delta^3$, which is the required volume.

EXAMPLES III (F)

Type 1. Ex. 1. Find the area of the triangle the coordinates of whose vertices are (1, 2, 3), (−2, 1, −4) and (3, 4, −2).

(Punjab T. D. C., 1965 S)

Sol. The coordinates of the points of projection in the *yz*-plane, of the three vertices are (0, 2, 3), (0, 1, −4) and (0, 4, −2). In two dimensions, these points are (2, 3), (1, −4) and (4, −2).

A_x=Area of the triangle formed by these points

$$=\frac{1}{2}\begin{vmatrix} 2 & 3 & 1 \\ 1 & -4 & 1 \\ 4 & -2 & 1 \end{vmatrix}=\frac{19}{2}.$$

Similarly, $A_y=\frac{29}{2}$, $A_z=-2$.

$$\therefore\ A=\sqrt{A_x^2+A_y^2+A_z^2}=\sqrt{\left(\frac{19}{2}\right)^2+\left(\frac{29}{2}\right)^2+4}=\frac{1}{2}\sqrt{1218}.$$

Aliter.

Let A(1, 2, 3), B(−2, 1, −4) and C(3, 4, −2) be the vertices of the triangle ABC.

---

* Let $\Delta'=\begin{vmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{vmatrix}$

$$\therefore\ \Delta\Delta'=\begin{vmatrix} \Delta & 0 & 0 & 0 \\ 0 & \Delta & 0 & 0 \\ 0 & 0 & \Delta & 0 \\ 0 & 0 & 0 & \Delta \end{vmatrix}$$

$$=\Delta^4,$$

i. e., $\Delta'=\Delta^3$.

Let **i**, **j**, **k** be the unit vectors along the positive axes of *x*, *y* and *z* of a right-handed system.

$$\therefore\ \vec{OA}=\mathbf{i}+2\mathbf{j}+3\mathbf{k},\quad \vec{OB}=-2\mathbf{i}+\mathbf{j}-4\mathbf{k}.$$

$$\vec{OC}=3\mathbf{i}+4\mathbf{j}-2\mathbf{k}.$$

$\therefore\quad \vec{AB}=\vec{OB}-\vec{OA}=-3\mathbf{i}-\mathbf{j}-7\mathbf{k},$

$\vec{AC}=\vec{OC}-\vec{OA}=2\mathbf{i}+2\mathbf{j}-5\mathbf{k}.$

Now, area of the $\triangle ABC=\frac{1}{2}[\vec{AB}\times\vec{AC}]$

$=\frac{1}{2}[(-3\mathbf{i}-\mathbf{j}-7\mathbf{k})\times(2\mathbf{i}+2\mathbf{j}-5\mathbf{k})]$

$=\frac{1}{2}[-6\mathbf{k}-15\mathbf{j}+2\mathbf{k}+5\mathbf{i}-14\mathbf{j}+14\mathbf{i}]$

$=\frac{1}{2}[19\mathbf{i}-29\mathbf{j}-4\mathbf{k}]$

$\therefore$ area of $\triangle ABC=\frac{1}{2}\mid\vec{AB}\times\vec{AC}\mid=\frac{1}{2}\sqrt{1218}.$

Ex. 2. Find the areas of the triangles, whose vertices are the points :

(*i*) (4, 3, −2), (3, 0, 1) and (2, −1, 3). (*Poona, 1955*)

(*ii*) (2, −1, 3), (4, 3, −2), (3, 0, −1). (*Bombay, 1952*)

(*iii*) (1, 1, 3), (4, 3, 2), (5, 2, 7). [**Ans.** $\frac{1}{2}\sqrt{14}$; 5·79 nearly; $\frac{1}{2}\sqrt{362}$.] (*Bombay, 1954*)

Ex. 3. *Through a point P, (x', y', z'), a plane is drawn at right angles to OP to meet the axes (rectangular) in A, B, C. Prove that the area of the triangle ABC is $r^5/2x'y'z'$, where r is the measure of OP.*

(*Punjab, 1963 ; Vikram Engg., 1960 ; Roorkee, 1958*)

Sol. The direction ratios of OP are x', y', z'. The equation of the plane at right angles to OP is

$$xx'+yy'+zz'=\lambda \qquad \text{...(1)}$$

$\because$ it passes through P, $\quad\therefore\ \lambda=x'^2+y'^2+z'^2$

$\therefore$ (1) becomes $\quad xx'+yy'+zz'=x'^2+y'^2+z'^2 \qquad$ ...(2)

Let OP be r. $\quad\therefore\ x'^2+y'^2+z'^2=r^2.$

$\therefore$ (2) becomes $\quad xx'+yy'+zz'=r^2 \qquad$ ...(3)

$\therefore$ Co-ordinates of A, B, C are respectively

$$\left(\frac{r^2}{x'}, 0, 0\right), \left(0, \frac{r^2}{y'}, 0\right), \left(0, 0, \frac{r^2}{z'}\right).$$

$\therefore\ \triangle_x$ = area of the projection of the $\triangle ABC$ on the yz-plane

$$=\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ \frac{r^2}{y'} & 0 & 1 \\ 0 & \frac{r^2}{z'} & 1 \end{vmatrix}=\frac{1}{2}\cdot\frac{r^4}{y'z'}.$$

Similarly, $\triangle_y=\frac{1}{2}\frac{r^4}{z'x'}$ and $\triangle_z=\frac{1}{2}\cdot\frac{r^4}{x'y'}$,

Now, $\Delta^2=\Delta_x^2+\Delta_y^2+\Delta_z^2$

$$=\frac{1}{4}.r^8\left(\frac{1}{y'^2z'^2}+\frac{1}{z'^2x'^2}+\frac{1}{x'^2y'^2}\right)$$

$$=\frac{1}{4}.r^8\left(\frac{x'^2+y'^2+z'^2}{x'^2y'^2z'^2}\right)=\frac{1}{4}\frac{r^{10}}{x'^2y'^2z'^2}$$

$$\therefore \quad \Delta=\frac{1}{2}\cdot\frac{r^5}{x'y'z'}.$$

Ex. 4. Find the area of the triangle included between the plane

$$2x-3y+4z=12$$

and the coordinate planes. ***(Gujarat, 1958)***

[Ans. $3\sqrt{29}$.]

Type II. Ex. 1. A, B, C are the points (3, 2, 1), (−2, 0, −3) (0, 0, −2). Find the locus of P, if the volume PABC is 5. ***(Poona, 1958)***

Sol. Let (x, y, z) be the coordinates of P.

$\therefore$ by the problem, 5=volume PABC

or, $5=-\frac{1}{6}\begin{vmatrix} x & y & z & 1 \\ 3 & 2 & 1 & 1 \\ -2 & 0 & -3 & 1 \\ 0 & 0 & -2 & 1 \end{vmatrix}=-\frac{1}{6}\begin{vmatrix} x & y & z+2 & 0 \\ 3 & 2 & 3 & 0 \\ -2 & 0 & -1 & 0 \\ 0 & 0 & -2 & 1 \end{vmatrix},$

on subtracting the fourth row in succession from first, second and third rows.

or, $30=-\begin{vmatrix} x & y & z+2 \\ 3 & 2 & 3 \\ -2 & 0 & -1 \end{vmatrix}=-[x(-2)-y(-3+6)+(z+2)(0+4)],$

or, $2x+3y-4z=38$, which is the required equation of the locus of P.

Ex. 2. Find the volume of the tetrahedron formed by the points :

(*i*) (1, 1, 3), (4, 3, 2), (5, 2, 7) and (6, 4, 8). ***(Bombay, 1954)***

(*ii*) (1, 0, 0), (0, 0, 1), (0, 0, 2) and (1, 2, 3). ***(Gujarat, 1954)***

(*iii*) (1, 2, 3), (2, 3, 5), (−2, −1, 2), (3, 0, −3).

$\left[\textbf{Ans. } \frac{14}{3}\ ;\ \frac{1}{3}\ ;\ 3\frac{1}{3}\ .\right]$

Type III. Ex. 1. Find the volume of the tetrahedron formed by the planes whose equations are

$2x+3y+z=6$, $2x+3y=0$, $3y+z=0$ and $2x+z=0$. ***(Raj. Engg., 1953)***

Sol. The given planes are

$2x+3y+z=6$ ...(1), $\quad 2x+3y=0$ ...(2),

$3y+z=0$ ...(3), and $\quad 2x+z=0$ ...(4).

Solving (1), (2) and (3), $x=3$, $y=-2$, $z=6$.

Solving (1), (2) and (4), $x=3$, $y=2$, $z=-6$.

Solving (1), (3) and (4), $x=-3$, $y=2$, $z=6$.

Solving (2), (3) and (4), $x=0, y=0, z=0$.

$\therefore$ the vertices of the tetrahedron are $(3, -2, 6), (3, 2, -6), (-3, 2, 6)$ and $(0, 0, 0)$.

Let V be the volume required.

$$\therefore \quad V=\frac{1}{6}\begin{vmatrix} -3 & 2 & 6 & 1 \\ 3 & -2 & 6 & 1 \\ 3 & 2 & -6 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \frac{1}{6}\begin{vmatrix} -3 & 2 & 6 \\ 3 & -2 & 6 \\ 3 & 2 & -6 \end{vmatrix}$$

$$=\frac{1}{6}[-3(12-12)-2(-18-18)+6(6+6)]$$

$$=\frac{1}{6}[72+72]=\frac{144}{6}=24 \text{ cubic units.}$$

Ex. 2. Prove that the four planes $my+nz=0$, $nz+lx=0$, $lx+my=0$ and $lx+my+nz=p$, form a tetrahedron whose volume is $2p^3/3lmn$.

(Punjab, 1961 S ; Baroda, 1953)

Ex. 3. Find the volume of the tetrahedron formed by the planes $y+z=0$, $z+x=0$, $x+y=0$ and $x+y+z=1$.

(Delhi Hons., 1954 ; Punjab (Pakistan), 1954)

[Ans. $\frac{2}{3}$]

SECTION VII

SYSTEMS OF PLANES

3·22. To find the equation of any plane through the line of intersection of the planes $Ax+By+Cz+D=0$ and $A'x+B'y+C'z+D'=0$.

The equations of the given planes are

$$Ax+By+Cz+D=0 \qquad \text{...(1)}$$

and

$$A'x+B'y+C'z+D'=0 \qquad \text{...(2)}$$

Let us consider the equation

$$Ax+By+Cz+D+\lambda\,(A'x+B'y+C'z+D')=0 \qquad \text{...(3)}$$

It is a first degree equation in x, y and z.

$\therefore$ it represents a plane. (Art. 3·4)

The coordinates of the points which satisfy both (1) and (2) also satisfies (3) for all values of λ.

$\therefore$ (3) passes through all the points common to the planes (1) and (2).

Hence (3) is the equation of the plane which passes through the line of intersection of the planes (1) and (2).

Note. *In problems λ is determined from the additional condition provided in the problem.*

EXAMPLES III (G)

Ex. 1. The plane lx+my=0 is rotated about its line of intersection with the plane z=0 through an angle α. Prove that the equation to the plane in its new position is

$$lx+my\pm z\sqrt{l^2+m^2}\tan\alpha=0.$$

(*Agra, 1961* ; *A.M.I.E., Nov. 1962* ; *Bihar, 1962*)

Sol. Let the equation of any plane through the line of intersection of the planes $lx+my=0$ ...(1)

and $z=0$ ...(2)

be $lx+my+\lambda z=0$ ...(3)

$\because$ (3) makes an angle α with (1),

$\therefore$ $\cos\alpha=(l^2+m^2)/\sqrt{l^2+m^2}\sqrt{l^2+m^2+\lambda^2}$,

or, $\cos^2\alpha=\dfrac{l^2+m^2}{l^2+m^2+\lambda^2}$, or, $\lambda^2\cos^2\alpha=(l^2+m^2)\sin^2\alpha$.

$\therefore$ $\lambda=\pm\sqrt{(l^2+m^2)}.\tan\alpha$

$\therefore$ (3) becomes $lx+my\pm z\sqrt{l^2+m^2}.\tan\alpha=0$, which is the required equation.

Ex. 2. Find the equations to the line through (f, g, h) which is parallel to the plane $lx+my+nz=0$ and intersects the line $ax+by+cz+d=0$, $a'x+b'y+c'z+d'=0$, (*Delhi Hons., 1954*)

[**Ans.** $lx+my+nz=lf+mg+nh$, $(ax+by+cz+d)/(af+bg+ch+d)$
$=\dfrac{(a'x+b'y+c'z+d')}{(a'f+b'g+c'h+d')}$]

[**Hint.** *The required line is the line of intersection of the (i) plane parallel to the plane $lx+my+nz=0$ and passing through (f, g, h) and (ii) plane through the line $ax+by+cz+d=0$, $a'x+b'y+c'z+d'=0$ and passing through (f, g, h).*]

Ex. 3. Find the equation to the plane through the line

$$u\equiv ax+by+cz+d=0,$$
$$v=a'x+b'y+c'z+d'=0$$

parallel to the line $\dfrac{x}{l}=\dfrac{y}{m}=\dfrac{z}{n}$.

(*Delhi Hons., 1957* ; *Lucknow 1953* ; *Peshawar, 1955*)

Ex. 4. Prove that the plane through the point (α, β, γ) and the line $x=py+q=rz+s$, is given by

$$\begin{vmatrix} x & py+q & rz+s \\ \alpha & p\beta+q & r\gamma+s \\ 1 & 1 & 1 \end{vmatrix}=0.$$

(*Delhi Hons., 1955* ; *Pakistan (Punjab), 1952 S* ; *Raj., 1951*)

MISCELLANEOUS EXAMPLES ON CHAPTER III

1. Show that the point $(-1, 4, -3)$ is the circumcentre of the triangle fermed by the points $(3, 2, -5)$, $(-3, 8, -5)$ and $(-3, 2, 1)$. *(Patna, 1961)*

2. Prove that the planes $7x+4y-4z+33=0$, $36x-51y+12z+17=0$, $14x+8y-8z-12=0$ and $12x-17y+4z-3=0$ form the four faces of a rectangular parallelopiped. *(Lucknow, 1959)*

3. Find the equation of the plane which is perpendicular to the plane $4x+5y-3z=8$, and passes through the line of intersection of the planes $x+y+z=6$ and $2x+3y+4z+5=0$. *(Lucknow (Pass), 1963)*

[**Ans.** $x+7y+13z+96=0$.]

4. Find the equation to the plane through $(2, -3, -1)$ normal to the line joining $(3, 4, -1)$, $(2, -1, 5)$. *(Jodhpur Engg., 1963)*

[**Ans.** $x+5y-6z+7=0$].

5. Find the equation of the plane

(*i*) through the point $(-1, 3, 2)$ and perpendicular to the planes $x+2y+3z=5$, $3x+3y+z=9$. *(Magadh, 1964)*

(*ii*) through $(2, -1, 1)$ and the line $4x-3y+5=0$, $y-2z-5=0$. *(Patna, 1963)*

(*iii*) through the intersection of $2x+3y+10z=8$, $2x-3y+7z=2$ and perpendicular to $3x-2y+4z=5$. *(Bhagalpur, 1963)*

[**Ans.** (*i*) $7x-8y+3z+25=0$; (*ii*) $4x-y-4z=5$; (*iii*) $2y+z=2$.]

6. Find the equations of the planes through the points $(0, 4, -3)$, $(6, -4, 3)$ other than the plane through the origin, which cut off from the axes intercepts whose sum is zero. *(Patna, 1963)*

$$\left[\textbf{Ans.}\quad \frac{x}{3}-\frac{y}{2}-z=1 \text{ or } \frac{x}{3}+\frac{y}{6}-\frac{z}{9}=1.\right]$$

4

The Straight Line

SECTION 1

FORMS OF THE EQUATIONS OF LINES

4·1. Non-symmetrical form of the equation of a straight line.

The two planes $ax+by+cz+d=0$ and $a'x+b'y+c'z+d'=0$ taken **together** represent the line of intersection of the given planes.*

$\therefore$ a line can be determined as the intersection of two planes passing through it.

Note. *x-axis is the line of intersection of the planes xz, and xy.* $\therefore$ *its equations are* $y=0$, $z=0$ *taken together.*

Similarly, the equations of the y and z axes are respectively $x=0$, $z=0$ *and* $x=0$, $y=0$.

4·2. Symmetrical form of the equations of a straight line. To find the equations of the straight line passing through the point (x_1, y_1, z_1), and having direction cosines, l, m, n.

Let AB be the straight line passing through the point $A(x_1, y_1, z_1)$

Let $P(\lambda, \mu, \nu)$ be **any** point on AB.

Let AP be r.

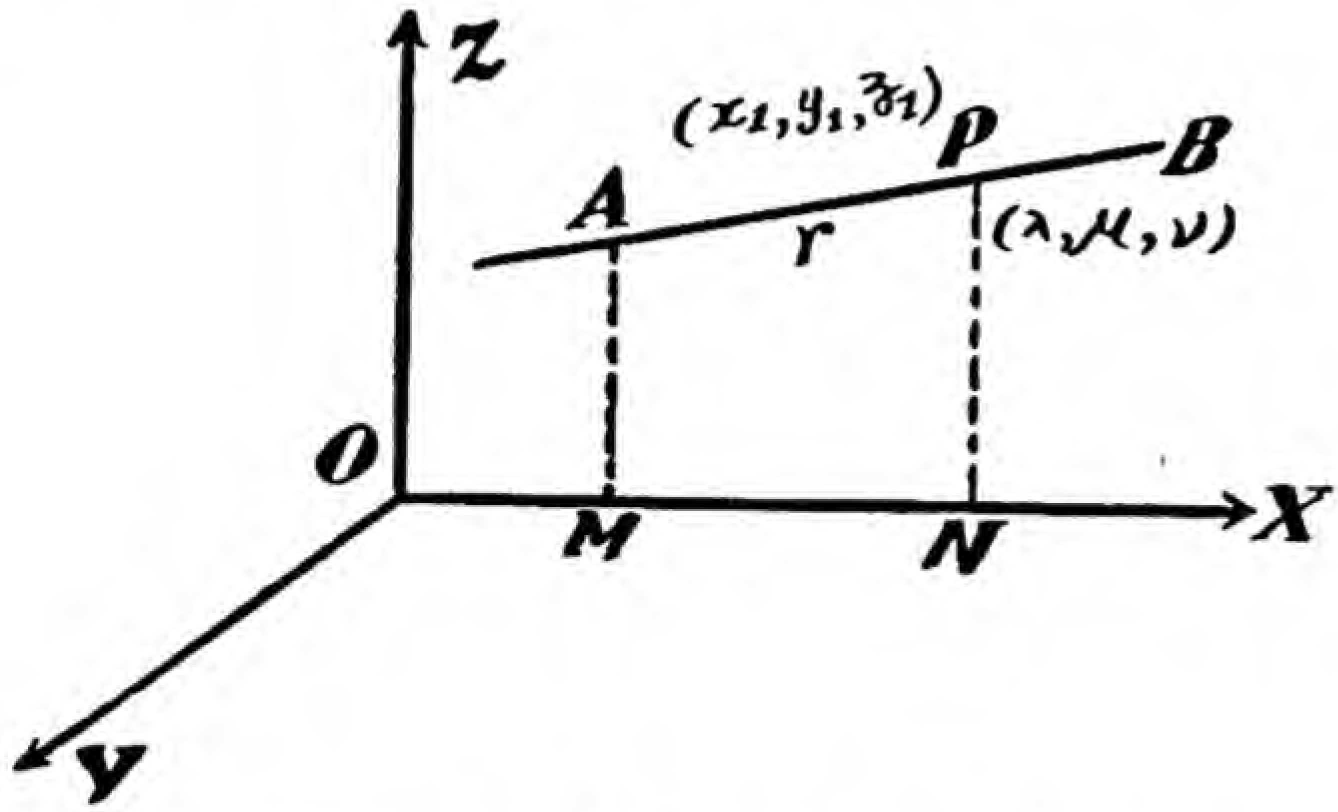
Let M and N be the feet of the perpendiculars from A and P respectively on the x-axis.

---

* $\because$ any point on the line of intersection of the planes $ax+by+cz+d=0$ and $a'x+b'y+c'z+d'=0$ lies on both these planes, $\therefore$ its coordinates satisfy both these equations. Conversely, any point whose coordinates satisfy the two equations of the planes lies on both these planes and therefore on the line of their intersection.

$\therefore$ $OM=x_1$ and $ON=\lambda$.

$\therefore$ $MN=ON-OM=\lambda-x_1$.



Now MN=projection of AP on the x-axis

$=r\cos\alpha$, where α is the angle which AB makes with the x-axis,

or $\lambda-x_1=r\cos\alpha=rl$.

Similarly, $\mu-y_1=rm$ and $\nu-z_1=rn$.

$$\therefore\quad \frac{\lambda-x_1}{l}=\frac{\mu-y_1}{m}=\frac{\nu-z_1}{n}=r.$$

$\therefore$ locus of (λ, μ, ν) is

$$\mathbf{\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r.}$$

which are the required equations of AB.

Note. *The vector equation of the line corresponding to the equation*

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

is $\mathbf{r}=\mathbf{a}+t\mathbf{b}$, *where* $\mathbf{a}=x_1\,\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k}$ *and* $\mathbf{b}=l\,\mathbf{i}+m\mathbf{j}+n\mathbf{k}$.

Cor. 1. Coordinates of any point on the line in terms of the parameter r.

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r \qquad \ldots(1)$$

$\therefore$ $x=x_1+lr,\ y=y_1+mr,\ z=z_1+nr.$

$\therefore$ the coordinates of any point on the line (1) are

$$(\mathbf{x_1+lr,\ y_1+mr,\ z_1+nr}).$$

Cor. 2. Symmetrical form of the equations of a straight line when its direction ratios are given.

Let a, b, c be the direction ratios.

$\therefore$ direction cosines are

$$\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}.$$

$\therefore$ from Art. 4·2, the equations of the line are

$$\frac{x-x_1}{a/\sqrt{a^2+b^2+c^2}}=\frac{y-y_1}{b/\sqrt{a^2+b^2+c^2}}=\frac{z-z_1}{c/\sqrt{a^2+b^2+c^2}}.$$

or,
$$\frac{\mathbf{x}-\mathbf{x}_1}{\mathbf{a}}=\frac{\mathbf{y}-\mathbf{y}_1}{\mathbf{b}}=\frac{\mathbf{z}-\mathbf{z}_1}{\mathbf{c}}.$$

Note 1. *Any point on the line* $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}=R$ *is* $(x_1+aR,\ y_1+bR,\ z_1+cR)$.

Note 2. Very imponrtant.

The symmetrical equations of a straight line are of the same orm whethe the direction cosiues are used or the direction ratios are used. But if the direction cosines are used, then r is the distance of the point $P(x_1+lr, y_1+mr, z_1+nr)$ *from the point* (x_1, y_1, z_1) *and if the direction ratios are used, then this point* (x_1+aR, y_1+bR, z_1+cR) *is* **not** *at a distancc R from the point* (x_1, y_1, z_1).

4·3. Double point form of the equations of the straight line. To find the equations of the line through two points $(\mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1)$ and $(\mathbf{x}_2, \mathbf{y}_2, \mathbf{z}_2)$.

Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the position vectors of the points A (x_1, y_1, z_2) and B (x_2, y_2, z_2) respectively. Let $\mathbf{r}$ be the position vector of **any** point P on AB.

$$\therefore \quad \mathbf{r}_1+\overrightarrow{AP}=\overrightarrow{OP}=\mathbf{r}, \quad \text{or,} \quad \overrightarrow{AP}=\mathbf{r}-\mathbf{r}_1 \quad \text{...(1)}$$

$$\text{Also,} \quad \mathbf{r}_1+\overrightarrow{AB}=\overrightarrow{OB}=\mathbf{r}_2, \quad \text{or,} \quad \overrightarrow{AB}=\mathbf{r}_2-\mathbf{r}_1 \quad \text{...(2)}$$

$\because$ AP and AB are parallel, $\therefore$ $\overrightarrow{AP}=t\,.\,\overrightarrow{AB}$, where t is **any** scalar number.

$$\therefore \quad \mathbf{r}-\mathbf{r}_1=t(\mathbf{r}_2-\mathbf{r}_1) \quad \text{...(3)}$$

Let P be (x, y, z).

$\therefore$ $\mathbf{r}=\overrightarrow{OP}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the unit vectors of a right-handed system of axes.

Also, $\mathbf{r}_1=x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k}$ and $\mathbf{r}_2=x_2\mathbf{i}+y_2\mathbf{j}+z_2\mathbf{k}$.

$\therefore$ (3) gives $(x-x_1)\mathbf{i}+(y-y_1)\mathbf{j}+(z-z_1)\mathbf{k}=t[(x_2-x_3)\mathbf{i}$
$+(y_2-y_3)\mathbf{j}+(z_2-z_3)\mathbf{k}]$

$\because$ **i, j, k** are non-coplanar vectors,

$\therefore$ equating coefficient of like vectors, we have

$x-x_1=t(x_2-x_1)$, $y-y_1=t(y_2-y_1)$ and $z-z_1=t(z_2-z_1)$.

Eliminating t, we have

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1},$$

which are the required equations of AB.

Aliter.

The direction ratios of the line joining the given points are

$$x_2-x_1,\ y_2-y_1,\ z_2-z_1.$$

$\because$ the line passes through (x_1, y_1, z_1),

$\therefore$ its equations are

$$\frac{\mathbf{x}-\mathbf{x}_1}{\mathbf{x}_2-\mathbf{x}_1}=\frac{\mathbf{y}-\mathbf{y}_1}{\mathbf{y}_2-\mathbf{y}_1}=\frac{\mathbf{z}-\mathbf{z}_1}{\mathbf{z}_2-\mathbf{z}_1}.$$

EXAMPLES IV (A)

Type I. Ex. 1. Find the distance of the point $(-1, -5, -10)$ from the point of intersection of the line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}$ and the plane $x+y+z=5$. *(Patna, 1963 S ; Magadh, 1964 ; Ranchi, 1962 S)*

Sol. Let $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}=\text{R}.$

$\therefore$ any point on this line is $(3\text{R}+2, 4\text{R}-1, 12\text{R}+2)$

If it lies on the plane $x-y+z=5$, then

$3\text{R}+2-4\text{R}+1+12\text{R}+2=5$, or, $\text{R}=0$.

$\therefore$ point of intersection is $(2, -1, 2)$.

$\therefore$ required distance $=\sqrt{(2+1)^2+(-1+5)^2+(2+10)^2}=13.$

Ex. 2. Find the coordinates of the point of intersection of the line $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{-1}$ with the plane $2x+y+z=1$.

[**Ans.** $(-1, -2, 5)$.]

Ex. 3. Find the ratio in which the xy-plane divides the join of $(-3, 4, -8)$ and $(5, -6, 4)$.
Also, obtain the point of intersection of the line with the plane.

(Jodhpur Engg., 1965 S)

[**Ans.** $2:1$; $(7/3, -8/3, 0)$.]

Ex. 4. Find the distance of the point $(1, -2, 3)$ from the plane $x-y+z=5$ measured parallel to the line $\frac{x}{2}=\frac{y}{3}=\frac{z}{-6}$.

(Pakistan (Punjab), 1957 S ; Burdwan, 1964 ; Utkal, 1964)

Sol. Equations of the line through (1, —2, 3) and parallel to

$\frac{x}{2}=\frac{y}{3}=\frac{z}{-6}$ are $\frac{x-1}{2}=\frac{y+2}{3}=\frac{z-3}{-6}=\text{R}$, say.

Any point on the line is (1+2R, —2+3R, 3—6R).

If it lies on the plane, then

$(1+2\text{R})-(-2+3\text{R})+(3-6\text{R})=5$, or, $\text{R}=\frac{4}{7}$.

$\therefore$ the point is $\left(\frac{9}{7}, \frac{-11}{7}, \frac{15}{7}\right)$.

$\therefore$ required distance$=\sqrt{\frac{4}{49}+\frac{9}{49}+\frac{36}{49}}=1$.

Ex. 5. Find the coordinates of the points in which the line

$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ meets the coordinate planes.

(Calcutta, 1961)

$\left[\textbf{Ans. } \left(o, \beta-\frac{m\alpha}{l}, \gamma-\frac{n\alpha}{l}\right); \left(\alpha-\frac{l\beta}{m}, 0, \gamma-\frac{n\beta}{n}\right); \left(\alpha-\frac{l\gamma}{n}, \beta-\frac{m\gamma}{n}, 0.\right)\right]$

Type II (A). Ex. 1. Find the coordinates of the reflection of the point (2, —1, 3) in the plane 3x—2y+z—9=0. *(A.M.I.E., 1960)*

Sol. Let A be the point (2, —1, 3).

Draw AM perpendicular to the given plane. Produce AM to B such that AB=BM. Then B is the reflection of A in the plane.

Direction ratios of AM are 3, —2, —1.

$\therefore$ equations of the line AM are

$\frac{x-2}{3}=\frac{y+1}{-2}=\frac{z-3}{-1}=\text{R}$, say.

$\therefore$ any point on AM is (2+3R, —1—2R, —3—R).

If it lies on the plane $3x-2y-z-9=0$,

$3(2+3\text{R})-2(-1-2\text{R})-(3-\text{R})-9=0,$

or, $\text{R}=\frac{2}{7}$.

$\therefore$ coordinates of M are $\left(\frac{20}{7}, \frac{-11}{7}, \frac{19}{7}\right)$.

Let B be (x_1, y_1, z_1)

$\therefore \quad \frac{20}{7}=\frac{x_1+2}{2}, \frac{-11}{7}=\frac{y_1-1}{2}, \frac{19}{7}=\frac{z_1+3}{2}.$

$\therefore \quad x_1=\frac{26}{7}, y_1=-\frac{15}{7}, z_1=\frac{17}{7}.$

$\therefore$ coordinates of B, the reflection of (1) in the given plane are

$\left(\frac{26}{7}, \frac{-15}{7}, \frac{17}{7}\right)$.

Note. *We can also find the coordinates of B as follows :*
The coordinates of M as above are $\left(\frac{20}{7}, \frac{-11}{7}, \frac{19}{7}\right)$.

$\because$ *B divides AM externally in the ratio 2 : 1,*

$\therefore$ *coordinates of B are*

$$\left[\frac{2 \cdot \frac{20}{7} - 1 \cdot 2}{2-1}, \frac{2 \cdot \left(-\frac{11}{7}\right)+1}{2-1}, \frac{2\left(\frac{19}{7}\right)-1}{2-1}\right]$$

or $\quad$ *are* $\left[\frac{26}{7}, \frac{-15}{7}, \frac{17}{7}\right]$.

Ex. 2. Find the image of the point

(*i*) (3, 5, 7) in the plane $2x+y+z=6$, (*Poona, 1959*)

(*ii*) $(-3, 8, 4)$ in the plane $6x-3y-2z+1=0$.

[**Ans.** $(-5, 1, 3)$; $(9, 2, 0)$].

(B) [Image of a line in a given plane.]

Note. (*i*) *If the line* $\frac{x-\alpha}{a}=\frac{y-\beta}{b}=\frac{z-\gamma}{c}$ *is parallel to the given plane, the image of this line in the plane is a straight line through the image of* (α, β, γ) *in the plane, parallel to the line itself.*

(*ii*) *If the given line* $\frac{x-\alpha}{a}=\frac{y-\beta}{b}=\frac{z-\gamma}{c}$ *is not parallel to the plane, the image of this line in the plane is the straight line joining the point in which the line meets the plane to the image of* (α, β, γ) *in the plane.*

Ex. 1. Find the equations of the image of the line $\frac{x-1}{3}=\frac{y-3}{5}=\frac{z-4}{2}$ **in the plane** $2x-y+z+3=0$. (*Karnatak, 1957*)

Sol. The given line is $\frac{x-1}{3}=\frac{y-3}{5}=\frac{z-4}{2}=R$, say.

Any point on this line is $(1+3R, 3+5R, 4+2R)$.

If it lies on the plane $2x-y+z+3=0$, then
$2+6R-3-5R+4+2R+3=0$, $\quad$ or, $R=-2$.

$\therefore$ the point in which the given line meets the given plane is $(-5, -7, 0)$.

Again, the line through (1, 3, 4) perpendicular to the given plane is

$$\frac{x-1}{2}=\frac{y-3}{-1}=\frac{z-4}{1}=r_1, \text{ say.} \qquad ...(1)$$

Any point on this line is $(1+2r_1, 3-r_1, 4+r_1)$.

$\therefore \quad 2+4r_1-3+r_1+4+r_1+3=0 \qquad \therefore \; r_1=-1.$

$\therefore$ (1) meets the given plane in the point $(-1, 4, 3)$.

Let (x_1, y_1, z_1) be the image of (1, 3, 4) in the given plane.

$\therefore \quad \frac{x_1+1}{2}=-1, \; \frac{y_1+3}{2}=4, \; \frac{z_1+4}{2}=3,$

or, $\quad x_1=-3,\ y_1=5,\ z_1=2.$

$\therefore$ the image of $(1, 3, 4)$ in the given plane is $(-3, 5, 2)$.

The image of the given line in the given plane is the line joining the points $(-5, -7, 0)$ and $(-3, 5, 2)$.

$\therefore$ its equations are

$$\frac{x+3}{-2}=\frac{y-5}{-12}=\frac{z-2}{-2},$$

or,
$$\frac{x+3}{1}=\frac{y-5}{6}=\frac{z-2}{1}.$$

Ex. 2. Find the image of the line $\dfrac{x+1}{1}=\dfrac{y+3}{3}=\dfrac{z-2}{-2}$ in the plane $3x+4y+5z=5$.

$$\left[\textbf{Ans. } \frac{x-\frac{1}{5}}{2}=\frac{y+\frac{7}{5}}{11}=\frac{z-4}{-15}.\right]$$

Ex. 3. Find the image of the line $\dfrac{x-1}{9}=\dfrac{y-2}{-1}=\dfrac{z+3}{-3}$ in the plane $3x-3y+10z=26$.

$$\left[\textbf{Ans. } \frac{x-4}{9}=\frac{y+1}{-1}=\frac{z-7}{-3}.\right]$$

Type III. Ex. 1. Find the coordinates of the foot of the perpendicular drawn from the point (5, 9, 3) to the line.

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}.$$

Sol. Any point on the given line is $(1+2r, 2+3r, 3+4r)$.

Let M be the foot of the perpendicular from $P(5, 9, 3)$ to the given line.

If $(1+2r, 2+3r, 3+4r)$ are the coordinates of M, then the direction ratios of PM are

$$1+2r-5,\ 2+3r-9,\ 3+4r-3, \quad \text{or,} \quad 2r-4,\ 3r-7,\ 4r.$$

$\because$ PM is perpendicular to the given line,

$\therefore \quad 2(2r-4)+3(3r-7)+4(4r)=0,$

or, $\quad 29r=29 \qquad \therefore\ r=1.$

$\therefore$ the coordinates of M are $(1+2, 2+3, 3+4)$, or $(3, 5, 7)$.

Ex. 2. From the point $P(1, 2, 3)$, PN is drawn perpendicular to the straight line $\dfrac{x-2}{3}=\dfrac{y-3}{4}=\dfrac{z-4}{5}$. Find the distance PN, the equations to PN and the coordinates of N. *(Raj. Engg., 1958)*

$$\left[\textbf{Ans. } \frac{1}{5}\sqrt{3}\ ;\ \frac{x-1}{7}=\frac{y-2}{1}=\frac{z-3}{-5}\ ;\ \left(\frac{32}{25}, \frac{51}{25}, \frac{14}{5}\right).\right]$$

Ex. 3. Find the coordinates of the foot of the perpendicular from $P(1, 4, 12)$ to the line $x+5=\dfrac{y+3}{4}=\dfrac{2z-6}{9}$. Deduce the equation of the perpendicular from P on the given line.

$$\left[\textbf{Ans. } (-3, 5, 12)\ ;\ \frac{x-1}{4}=\frac{y-4}{-1}=\frac{z-12}{0}.\right]$$

Type IV. Ex. 1. Prove that the points (4, 0, 4), (−2, 4, 2) (7, −2, 5) are collinear.

Sol. The equations of the line joining the points (4, 0, 4) and (−2, 4, 2)

are $\dfrac{x-4}{-2-4}=\dfrac{y-0}{4-0}=\dfrac{z-4}{2-4}$,

or, $\dfrac{x-4}{-6}=\dfrac{y}{4}=\dfrac{z-4}{-2}$, or, $\dfrac{x-4}{-3}=\dfrac{y}{2}=\dfrac{z-4}{-1}$ ...(1)

$\because$ (7, −2, 5) satisfies (1), $\therefore$ the points are collinear.

Ex. 2. Find the coordinates of the point in which the line joining (2, −3, 1) and (3, −4, 5) meets the plane $x+y+2z=13$.

[**Ans.** $(\frac{7}{2}, -\frac{9}{2}, 7)$.]

Ex. 3. Find the equations of the perpendicular from (3, −1, 11) to the line $\frac{1}{2}x=\frac{1}{3}(y-2)=\frac{1}{4}(z-3)$. *(Punjab, 1956S)*

$$\left[\textbf{Ans.}\ \frac{x-3}{1}=\frac{y+1}{-6}=\frac{z-11}{4}.\right]$$

Ex. 4. Find the equations to the line joining (2, 4, 3) and (−3, 5, 3), and show that this line is the intersection of the planes $x+5y=22$, $z=3$.

(Jodhpur Engi., 1963)

$$\left[\textbf{Ans.}\ \frac{x-2}{-5}=\frac{y-4}{1}=\frac{z-3}{0}.\right]$$

SECTION II

REDUCTION OF NON-SYMMETRICAL EQUATIONS OF A STRAIGHT LINE TO THE SYMMETRICAL FORM

4·4. To reduce the equations of a line, viz.,

$$\mathbf{ax+by+cz+d=0,}$$
$$\mathbf{a'x+b'y+c'z+d'=0}$$

into the symmetrical form. *(Punjab, 1962)*

The given equations of the line are

$$\left.\begin{aligned} ax+by+cz+d&=0,\\ a'x+b'y+c'z+d'&=0\end{aligned}\right\} \quad ...(1)$$

[**To find the direction-ratios of the line (1).**]

The equations of the line through the origin parallel to the given line (1) are

$$\left.\begin{aligned} ax+by+cz&=0\\ a'x+b'y+c'z&=0\end{aligned}\right\} \quad ...(2)$$

Now, (2) may be written as

$$\frac{x}{bc'-b'c}=\frac{y}{ca'-c'a}=\frac{z}{ab'-a'b}.$$

$\therefore$ **The direction-ratios of the given line are**

$bc'-b'c,\ ca'-c'a,\ ab'-a'b$.

[To find the coordinates of one point on the line (1).]

There are infinite number of points on (1) and we have to choose any one of these points. For the sake of convenience we may choose the point as the point of intersection of the line (1) with the plane $z=0$.

This point is given by the equations

$$ax+by+d=0,$$
$$a'x+b'y+d'=0,\ z=0$$

$\therefore$ the point is

$$\left(\frac{bd'-b'd}{ab'-a'b},\ \frac{da'-d'a}{ab'-a'b},\ 0\right)$$

$\therefore$ The equations of the line (1) in the symmetrical form are

$$\frac{x-\left(\dfrac{bd'-b'd}{ab'-a'b}\right)}{bc'-b'c}=\frac{y-\dfrac{da'-d'a}{ab'-a'b}}{ca'-c'a}=\frac{z}{ab'-a'b}.$$

Note. *For finding a point on the given line, it is not always necessary to make $z=0$. We may, if convenient, make x or y zero or even give some other suitable value to any of the three variables. This all depends upon the given equations of the problem.*

EXAMPLES IV (B)

Type I. Ex. 1. Put in symmetrical form, the equations of the line $3x+2y-z-4=0=4x+y-2z+3$, and find the direction cosines.

(*Punjab, 1953*)

Sol.

The direction ratios of the line are given by the equations

$$3x+2y-z=0 \text{ and } 4x+y-2z=0.$$

$\therefore$
$$\frac{x}{-4+1}=\frac{y}{-4+6}=\frac{z}{3-8},$$

or,
$$\frac{x}{3}=\frac{y}{-2}=\frac{z}{5}.$$

$\therefore$ the direction ratios of the given line are $3, -2, 5$.

To find a point on the given line, putting $z=0$ in the given equations, we have

$$3x+2y-4=0 \text{ and } 4x+y+3=0,\ z=0$$

$\therefore$
$$\frac{x}{6+4}=\frac{y}{-16-9}=\frac{1}{3-8},\ z=0,$$

or,
$$x=-2,\ y=5,\ z=0.$$

$\therefore$ any point on the given line is $(-2, 5, 0)$.

$\therefore$ equations of the line in the symmetrical form are

$$\frac{x+2}{3}=\frac{y-5}{-2}=\frac{z}{5}.$$

Ex. 2. Express the equations of the following lines in the symmetrical form :

(*i*) $x+y+z+1=0,\ 4x+y-2z+2=0.$

(*ii*) $4x+4y-5z=12,\ 8x+12y-13z=32$ [*Punjab (Pakistan), 1953*]

$$\left[\text{Ans.}\quad \frac{x+\frac{1}{3}}{1}=\frac{y+\frac{2}{3}}{-2}=\frac{z}{1}\ ;\quad \frac{x-1}{2}=\frac{y-2}{3}=\frac{z}{4}.\right]$$

Type II. Ex. 1. Prove that the lines $x+y-z=5,\ 9x-5y+z=4$ and $6x-8y+4z=3,\ x+8y-6z+7=0$ are parallel. (*Raj. Engi., 1960*)

Sol. The direction ratios of the first line are given by

$$x+y-z=0 \text{ and } 9x-5y+z=0.$$

$$\therefore \qquad \frac{x}{1-5}=\frac{y}{-9-1}=\frac{z}{-5-9},$$

or,
$$\frac{x}{-4}=\frac{y}{-10}=\frac{z}{-14},$$

or,
$$\frac{x}{2}=\frac{y}{5}=\frac{z}{7}.$$

$\therefore$ the direction ratios of the first line are 2, 5, 7.

The direction ratios of the second line are given by

$$6x-8y+4z=0 \text{ and } x+8y-6z=0$$

$$\therefore \qquad \frac{x}{48-32}=\frac{y}{4+36}=\frac{z}{48+8},$$

or,
$$\frac{x}{16}=\frac{y}{40}=\frac{z}{56},$$

16470.

or,
$$\frac{x}{2}=\frac{y}{5}=\frac{z}{7},$$

$\therefore$ the direction ratios of the second line are 2, 5, 7.

$\because$ the direction ratios of the lines are proportional,

$\therefore$ the two lines are parallel.

Ex. 2. Show that the line $2x+2y-z-6=0=2x+3y-z-8$ is parallel to the plane $y=0$, and find the coordinates of the point where it meets the plane $x=0$. (*Delhi Hons., 1953 ; Baroda, 1953*)

[**Ans.** $(0, 2, -2)$.]

Ex. 3. Show that the lines

$$3x+2y+z-5=0=x+y-2z-3$$

and
$$8x-4y-4z=0=7x+10y-8z \text{ are perpendicular.}$$

(*Bombay, 1954*)

Type III, Ex. 1. Find the equation to plane through the point (1, 2, 3) perpendicular to the line

$$x-y+2z=5,\ 3x+y+z=6.$$

(Karnatak, 1952 ; A.M.I E. May, 1955)

Sol. The direction ratios of the given line are given by

$$x-y+2z=0 \text{ and } 3x+y+z=0.$$

$$\therefore \qquad \frac{x}{-1-2}=\frac{y}{6-1}=\frac{z}{1+3}$$

or,
$$\frac{x}{-3}=\frac{y}{5}=\frac{z}{4}.$$

$\therefore$ direction ratios are $-3, 5, 4$.

$\because$ required plane is perpendicular to this line,

$\therefore$ normal to the plane is parallel to the line.

Let the plane be $Ax+By+Cz+D=0$ ...(1)

$$\therefore \qquad \frac{A}{-3}=\frac{B}{5}=\frac{C}{4} \qquad ..(2)$$

$\because$ (1) passes through (1, 2, 3), we have

$$A+2B+3C+D=0 \qquad ...(3)$$

Subtracting (3) from (1), we have

$$A(x-1)+B(y-2)+C(z-3)=0 \qquad ...(4)$$

From (2) and (4), we have

$$-3(x-1)+5(y-2)+4(z-3)=0,$$

or,
$$3x-5y-4z+19=0.$$

Ex. 2. The planes $3x-y+z+1=0$, $5x+y+3z=0$ intersects in the line PQ and a plane is drawn through the point (2, 1, 4) perpendicular to PQ. Show that this plane contains the point (2, 3, 5). *(Bombay, 1955)*

Ex. 3. Find the equation of the line through the point (1, 2, 3) parallel to the line $3x+y+z=6$, $x-y+2z=5$.

(Magadh, 1963, 1964; Punjab B.Sc., 1964 S)

$$\left[\ \textbf{Ans. } \frac{x-1}{-3}=\frac{y-2}{5}=\frac{z-3}{4}.\ \right]$$

Ex. 4. Find the equation of the line through the point (1, 3, 4) parallel to the line $x+2y+3z=8$, $2x+y-z=9$. *(Gauhati, 1962)*

$$\left[\ \textbf{Ans. } \frac{x-1}{5}=\frac{y-3}{-7}=\frac{z-4}{3}.\ \right]$$

SECTION III

CONSTANTS IN THE EQUATIONS OF A LINE

4·5. To show that the non-symmetrical equations of a straight line contains four arbitrary constants.

Proof. The equations of the straight line in the symmetrical form are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad \ldots(1)$$

From first and second members of (1), we have

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m},$$

or, $$x-\alpha=\frac{l}{m}(y-\beta),$$

or, $$x=\alpha+\frac{l}{m}(y-\beta)$$

or, $$x=\frac{l}{m}y+\left(\alpha-\frac{l}{m}\beta\right)=Ay+B,$$

where $$A=\frac{l}{m},\ B=\alpha-\frac{l}{m}\beta.$$

From the second and third members of (1), we have

$$\frac{y-\beta}{m}=\frac{z-\gamma}{n},$$

or, $$y=\beta+\frac{m}{n}(z-\gamma)=\frac{m}{n}z+\left(\beta-\frac{m}{n}\gamma\right),$$

or, $$y=Cz+D,$$

where $$C=\frac{m}{n} \text{ and } D=\beta-\frac{m}{n}\gamma.$$

$\therefore$ (1) can be written in the form

$x=Ay+B,\ y=Cz+D$, which contains four arbitrary constants.

$\therefore$ the equations of a line contain four arbitrary constants. This proves the proposition.

EXAMPLES IV (C)

Ex. 1. Prove that the lines $x=ay+b$, $z=cy+d$, and $x=a'y+b'$, $z=c'y+d'$ are perpendicular if $aa'+cc'+1=0$.

Sol. The given lines are

$$\frac{x-b}{a}=\frac{y}{1}=\frac{z-d}{c}$$

and
$$\frac{x-b'}{a'}=\frac{y}{1}=\frac{z-d'}{c'}.$$

If they are perpendicular then $aa'+cc'+1=0$.

Ex. 2. Find a, b, c, d so that the line $x=ay+b$, $z=cy+d$ may pass through the points $(3, 2, -4)$, $(5, 4, -6)$ and hence show that the given points and $(9, 8, -10)$ are collinear.

[**Ans.** $a=1, b=1, c=-1, d=-2.$]

Ex. 3. Find the equations of the line through the point $(1, 2, 3)$ parallel to the line $x=ay+b$, $z=cy+d$.

$\left[\textbf{Ans.}\quad \frac{x-1}{a}=\frac{y-2}{1}=\frac{z-3}{c}.\right]$

SECTION IV

A PLANE AND A LINE

4·6. Condition of parallelism and perpendicularity.

To find the conditions that the line

$$\frac{\mathbf{x}-\alpha}{\mathbf{l}}=\frac{\mathbf{y}-\beta}{\mathbf{m}}=\frac{\mathbf{z}-\gamma}{\mathbf{n}}$$

is (*i*) **parallel,** (*ii*) **perpendicular to the plane**

$$\mathbf{Ax+By+Cz+D=o}.$$

The given line is

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad \text{.. (1)}$$

and the given plane is

$$Ax+By+Cz+D=0 \qquad \text{...(2)}$$

(*i*) If (1) is parallel to (2), then

(*a*) (1) is perpendicular to the normal to the plane (2) and (*b*) the point (α, β, γ) does not lie on (2).

Now, the direction ratios of the line are l, m, n and that of the normal to plane (2) are A, B, C.

$\therefore$ (*a*) gives $\quad \mathbf{Al+Bm+Cn=o},$

and (*b*) gives $\quad \mathbf{A\alpha+B\beta+C\gamma+D\neq o},$

which are the required conditions.

(*ii*) If (1) is perpendicular to (2), then (1) is parallel to the normal to the plane (2).

$\therefore\ \frac{\mathbf{l}}{\mathbf{A}}=\frac{\mathbf{m}}{\mathbf{B}}=\frac{\mathbf{n}}{\mathbf{C}}$, which are the required conditions.

4·7. To find the conditions that the line $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ may lie in the plane $Ax+By+Cz+D=o$.

The equations of the given line are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad ...(1)$$

and the equation of the given plane is

$$Ax+By+Cz+D=0 \qquad ...(2)$$

If (1) lies in (2), then

(*i*) (1) is perpendicular to the normal to the plane (2) and (*ii*) the point (α, β, γ) lies on (2).

Now the direction ratios of (1) are l, m, n and that of the normal to the plane (2) are A, B, C.

$\therefore$ (*i*) gives $\qquad$ $\mathbf{Al+Bm+Cn=o}$,

and $\quad$ (*ii*) gives $\qquad$ $\mathbf{A\alpha+B\beta+C\gamma+D=o}$,

which are the required conditions.

4·8. Angle formula for a line and a plane.

To find the angle between the line $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ and the plane $Ax+By+Cz+D=o$.

The given line and the plane are respectively.

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad ...(1)$$

and

$$Ax+By+Cz+D=0. \qquad ...(2)$$

Let θ be the angle between (1) and (2).

$\therefore$ the angle between (1) and the normal to the plane (2) is $90°-\theta$.

The direction ratios of (1) are l, m, n and that of the normal to the plane are A, B, C.

$$\therefore \cos(90°-\theta)=\frac{Al+Bm+Cn}{\sqrt{l^2+m^2+n^2}\ \sqrt{A^2+B^2+C^2}},$$

or

$$\theta=\sin^{-1}\left[\frac{Al+Bm+Cn}{\sqrt{l^2+m^2+n^2}\ \sqrt{A^2+B^2+C^2}}\right].$$

which is the required angle.

4·9. To find the equation of any plane through the line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}.$$

Equation of any plane through (α, β, γ) is

$$A(x-\alpha)+B(y-\beta)+C(z-\gamma)=0 \qquad ...(1)$$

$\because$ (1) passes through the given line, the normal to (1) is perpendicular to the given line,

$$\therefore \qquad Al+Bm+Cn=0 \qquad ...(2)$$

Hence *the equation of any plane through the given line is*

$$\mathbf{A(x-\alpha)+B(y-\beta)+C(z-\gamma)=0,}$$

where $\qquad \mathbf{Al+Bm+Cn=0.}$

4·10. To find the equation of any plane through the line $\frac{x-\alpha}{l_1}=\frac{y-\beta}{m_1}=\frac{z-\gamma}{n_1}$ **and parallel to the line** $\frac{x}{l_2}=\frac{y}{m_2}=\frac{z}{n_2}$.

From Art. 4·9, the equation of any plane through the line

$$\frac{x-\alpha}{l_1}=\frac{y-\beta}{m_1}=\frac{z-\gamma}{n_1} \text{ is}$$

$$A(x-\alpha)+B(y-\beta)+C(z-\gamma)=0 \qquad ..(1)$$

where $\qquad Al_1+Bm_1+Cn_1=0 \qquad ...(2)$

$\because$ (1) is parallel to the line $\frac{x}{l_2}=\frac{y}{m_2}=\frac{z}{n_2}$,

$\therefore$ the normal to (1) is perpendicular to this line.

$$\therefore \quad Al_2+Bm_2+Cn_2=0 \qquad ...(3)$$

Eliminating A, B, C between (1), (2) and (3), the equation of the required plane is

$$\begin{vmatrix} x-\alpha & y-\beta & z-\gamma \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

EXAMPLES IV (D)

Type I. Ex. 1. Find the equation to the plane containing the line $\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z+2}{1}$ **and the point** $(0, 7, -7)$ **and show that the line** $\frac{x}{1}=\frac{y-7}{-3}=\frac{z+7}{2}$ **also lies in the same plane.** *(Punjab, 1960)*

Sol. The equation of any plane containing the line

$$\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z+2}{1}$$

is $A(x+1)+B(y-3)+C(z+2)=0$, ...(1)

where $$-3A+2B+C=0 \quad ...(2) \qquad \text{(Art. 4·9)}$$

$\because$ (1) passes through $(0, 7, -7)$ $\therefore$ $A+4B-5C=0$ ...(3)

From (2) and (3), we have

$$\frac{A}{-10-4}=\frac{B}{1-15}=\frac{C}{-12-2},$$

or, $$\frac{A}{1}=\frac{B}{1}=\frac{C}{1}.$$

$\therefore$ (1) becomes $(x+1)+(y-3)+(z+2)=0$,

or, $$x+y+z=0 \quad ...(4)$$

The direction ratios of the line

$$\frac{x}{1}=\frac{y-7}{-3}=\frac{z+7}{2} \quad ...(5)$$

are $1, -3, 2$ and that of the normal to (4) are $1, 1, 1$.

Now $1.1+1.(-3)+1.2=0$.

$\therefore$ line (5) is perpendicular to the normal to (4).

Also $(0, 7, -7)$ satisfies (4).

$\therefore$ line (5) lies in the plane (4). (Art. 4·7)

Ex. 2. Find the equation to the plane through the line

$$ax+by+cz=0=a'x+b'y+c'z,$$
$$\alpha x+\beta y+\gamma z=0=\alpha' x+\beta' y+\gamma' z.$$ *(Punjab, 1959 S)*

$$\left[\textbf{Ans.} \quad \begin{vmatrix} x & y & z \\ bc'-b'c & ca'-c'a & ab'-a'b \\ \beta\gamma'-\beta'\gamma & \gamma\alpha'-\gamma'\alpha & \alpha\beta'-\alpha'\beta \end{vmatrix}=0.\right]$$

Ex. 3. Show that the plane passing through the line

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$$

and perpendicular to the plane containing the lines

$$\frac{x}{m}=\frac{y}{n}=\frac{z}{l} \quad \text{and} \quad \frac{x}{n}=\frac{y}{l}=\frac{z}{m}$$

is $\Sigma(m-n)\,x=0$. *(Karnatak Engg., 1961 ; Lucknow, 1958 ; Pakistan (Punjab), 1954 S)*

Ex. 4. Show that the equation to the plane containing the line

$$\frac{y}{b}+\frac{z}{c}=1,\ x=0 \text{ and parallel to the line}$$

$$\frac{x}{a}-\frac{z}{c}=1,\ y=0 \text{ is } \frac{x}{a}-\frac{y}{b}-\frac{z}{c}+1=0.$$

(Karnatak, 1955)

Ex. 5. Show that the plane $5x-10y-6z-29=0$ contains the line

$$\frac{x-3}{4}=\frac{y+2}{-1}=\frac{z-1}{5}.$$ *(Raj. Engg., 1956)*

Ex. 6. Find the equation of the line through the point $(-2, 3, 4)$ and parallel to the planes $2x+3y+4z=5$ and $3x+4y+5z=6$.

(Jodhpur Engg., 1965 Sup.)

$$\left[\textbf{Ans.}\quad x-2=\frac{y-3}{-2}=z-4.\right]$$

Ex. 7. Obtain the equation of the plane through the line

$$\frac{x-1}{2}=\frac{y+2}{-3}=\frac{z}{5}$$

perpendicular to the plane $x-y+z+2=0$. *(Calcutta, 1961)*

[**Ans.** $2x+3y+z+4=0$]

Type II. Ex 1. Find the equations of the orthogonal projection of the line $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-4}{4}$ on the plane $x+3y+z+5=0$. *(Poona, 1960)*

Sol. The projection of the line on a plane is the line of intersection of the given plane and the plane through the line perpendicular to the given plane.

The given line is $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-4}{4}$ ...(1)

and the given plane is $x+3y+z+5=0$ .. (2)

Equation of any plane through (1) is

$A(x-1)+B(y-2)+C(z-4)=0$ ...(3)

where $2A+3B+4C=0$ ...(4)

If (3) is perpendicular to (2), then

$A+3B+C=0$ ...(5)

From (4) and (5), we have

$$\frac{A}{3-12}=\frac{B}{4-2}=\frac{C}{6-3},$$

or,
$$\frac{A}{-9}=\frac{B}{2}=\frac{C}{3}$$

$\therefore$ (3) becomes $-9(x-1)+2(y-2)+3(z-4)=0$

or, $-9x+9+2y-4+3z-12=0$,

or, $9x-2y-3z+7=0$ ...(6)

The equations of the required projection are

$$x+3y+z+5=0 \quad\text{and}\quad 9x-2y-3z+7=0.$$

Aliter.

If A be the point of intersection of the given line and the given plane, and B is the projection of any point of the line on the plane, then PQ is the projection of the given line on the given plane.

Any point on the given line is $(1+2R, 2+3R, 4+4R)$. If it lies on the given plane, then

$1+2R+6+9R+4+4R+5=0$,

or, $$15R=-16, \text{ or } R=-\tfrac{16}{15}.$$

$\therefore$ the point A where the given line meets the given plane is

$$\left[-\frac{17}{15},\ -\frac{6}{5},\ -\frac{4}{15}\right]$$

Any point on the given line is (1, 2, 4).

Equations of the line through (1, 2, 4) perpendicular to the given plane are

$$\frac{x-1}{1}=\frac{y-2}{3}=\frac{z-4}{1}.$$

Any point on this line is $(1+R_1, 2+3R_1, 4+R_1)$. If it lies on the given plane, then

$$1+R_1+6+9R_1+4+R_1+5=0,$$

or, $$11R_1=-16$$

$\therefore$ $$R_1=-\tfrac{16}{11}.$$

$\therefore$ the projection of the point (1, 2, 4) on the given plane is the point B

$$\left(-\frac{5}{11},\ -\frac{26}{11},\ \frac{28}{11}\right).$$

$\therefore$ equations of the line of projection are

$$\frac{x+\frac{17}{15}}{-\frac{5}{11}+\frac{17}{15}}=\frac{y+\frac{6}{5}}{-\frac{26}{11}+\frac{6}{5}}=\frac{z+\frac{4}{15}}{\frac{28}{11}+\frac{4}{15}}$$

or, $$\frac{x+\frac{17}{15}}{7}=\frac{y+\frac{6}{5}}{-12}=\frac{z+\frac{4}{15}}{29}.$$

Ex. 2. Find the projection of the line

(*i*) $\dfrac{x-1}{2}=\dfrac{y-2}{3}=\dfrac{z-4}{4}$ on the plane $3x+4y+5z=0$.

(*ii*) $3x-y+2z=1$, $x+2y-z-2=0$ on the plane $3x+2y+z=0$.

(*Punjab, 1956*)

(*iii*) $8x-y-7z-8=0$, $x+y+z-1=0$ on the plane $5x-4y-z=5$.

$$\left[\textbf{Ans.}\quad \frac{x}{7}=\frac{y}{1}=\frac{z}{-5};\ \frac{x+1}{11}=\frac{y-1}{-9}=\frac{z-1}{-15};\ \frac{x-1}{1}=\frac{y}{2}=\frac{z}{-3}.\right]$$

Ex. 3. Find the equation in the symmetrical form of the projection of the line $\dfrac{x-1}{2}=-(y+1)=\dfrac{z-3}{4}$ on the plane $x+2y+z=12$.

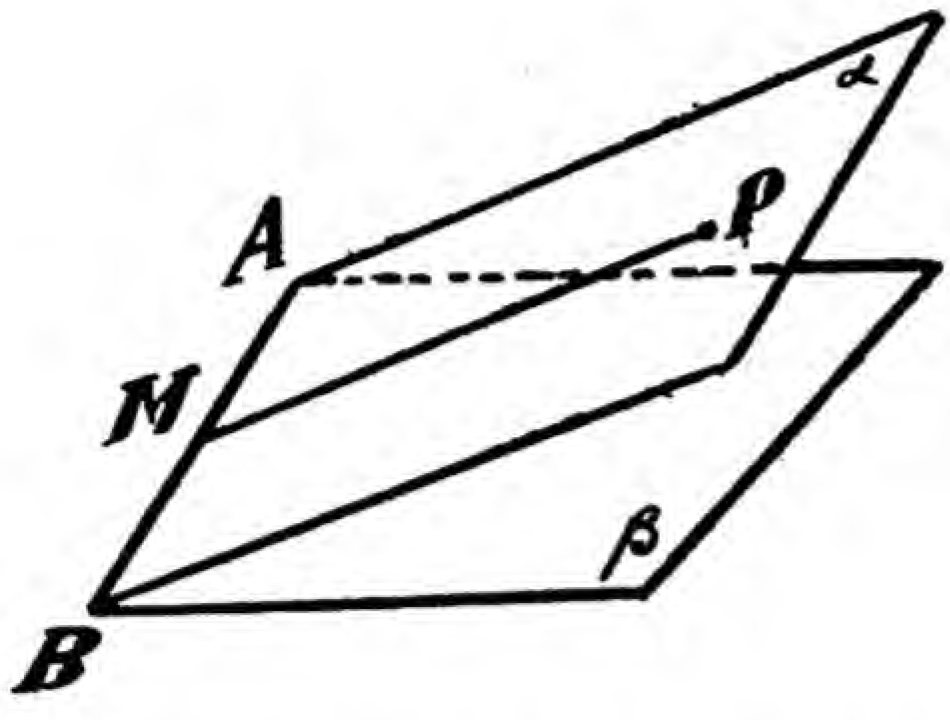
(*A.M.I.E., May, 1962*)

$$\left[\textbf{Ans.}\quad \frac{x-\frac{4}{5}}{4}=\frac{y-\frac{28}{5}}{-7}=\frac{z}{10}.\right]$$

Type III. [Equations of the line of greatest slope of a plane.]

[Note. Line of greatest slope : Def.

The line of greatest slope through a point on an inclined plane is the line (i) through the point, (ii) perpendicular to the line of intersection of the plane and a horizontal plane.



Let α be the given plane and β be a horizontal plane. Let AB be their line of intersection. Let P be any point on the plane α. Draw PM perpendicular to AB. Then PM is called the line of greatest slope on the plane α through the point P].

Ex. 1. Assuming the plane $4x-3y+7z=0$ to be horizontal, find (i) the equation of the vertical through the origin, and (ii) the direction cosines of the line of greatest slope in the plane $2x+y-5z=0$.

(A.M.I.E., 1957, 1961 ; Raj. Engg., 1959)

Also find the equations of the line of greatest slope through the point (0, 5, 1) on the plane $2x+y-5z=0$.

Sol. ∵ The vertical is a normal to the horizontal plane $4x-3y+7z=0$,

∴ the direction ratios of the vertical are 4, -3, 7.

∴ Equations of the vertical through the origin are

$$\frac{x}{4}=\frac{y}{-3}=\frac{z}{7}.$$

The direction ratios of the line of intersection AB of the planes

$4x-3y+7z=0$ and $2x+y-5z=0$ are given by

$4x-3y+7z=0$, $2x+y-5z=0$.

$$\therefore \qquad \frac{x}{4}=\frac{y}{17}=\frac{z}{5}.$$

∴ direction ratios of AB are 4, 17, 5.

Let l, m, n be the direction ratios of the line of greatest slope PM.

∵ PM lies in the plane $2x+y-5z=0$,

∴ $2l+m-5n=0$ ...(1)

Also, ∵ PM ⊥ AB, ∴ $4l+17m+5n=0$ ...(2)

From (1) and (2), we have

$$\frac{l}{90}=\frac{m}{-30}=\frac{n}{30}, \text{ or, } \frac{l}{3}=\frac{m}{-1}=\frac{n}{1},$$

$\therefore$ direction cosines of the line PM are

$$\frac{3}{\sqrt{11}}, \frac{-1}{\sqrt{11}}, \frac{1}{\sqrt{11}}.$$

The equations of the line of greatest slope through (0, 5, 1) are

$$\frac{x}{3}=\frac{y-5}{-1}=\frac{z-1}{1}.$$

Ex. 2. With given rectangular axes, the line

$$\frac{x}{2}=\frac{y}{-3}=\frac{z}{1}$$

is vertical. Find the direction cosines of the line of greatest slope in the plane $3x-2y+z=0$. *(A.M.I E., 1958, 1964 ; Agra Engg., 1962)*

$$\left[\textbf{Ans. } \frac{11}{3\sqrt{2}}, \frac{16}{3\sqrt{42}}, \frac{-1}{3\sqrt{42}}.\right]$$

[**Note.** *Any plane perpendicular to the vertical is horizontal.*

$\therefore$ *$2x-3y+z=0$ is the horizontal plane.*]

Ex. 3. Show that the plane $5x-10y-6z-29=0$ contains the line

$$\frac{x-3}{4}=\frac{y+2}{-1}=\frac{z-1}{5}. \qquad \textit{(Poona, 1953)}$$

Find the angle between this line and the line of greatest slope in the given plane, xy plane being assumed to be horizontal.

(Bombay, 1955 ; Poona, 1953 ; Raj. Engg., 1953)

$$\left[\textbf{Ans. } \cos^{-1}\left(\frac{161}{3\,[illegible]}\right).\right]$$

SECTION V

COPLANAR LINES—SHORTEST DISTANCE

4.11. Condition of coplanarity of lines. To find the condition that two straight lines

$$\frac{\mathbf{x}-\alpha}{\mathbf{l}}=\frac{\mathbf{y}-\beta}{\mathbf{m}}=\frac{\mathbf{z}-\gamma}{\mathbf{n}} \textbf{ and } \frac{\mathbf{x}-\alpha'}{\mathbf{l}'}=\frac{\mathbf{y}-\beta'}{\mathbf{m}'}=\frac{\mathbf{z}-\gamma'}{\mathbf{n}'}$$

may intersect (or are coplanar).

Using the note of Art. 4·2, the given equations of the lines are

$$\mathbf{r}=\mathbf{a}+t\mathbf{b} \quad \ldots(1) \qquad \text{and} \quad \mathbf{r}=\mathbf{a}'+s\mathbf{b}' \quad \ldots(2),$$

where $\mathbf{a}=\alpha\mathbf{i}+\beta\mathbf{j}+\gamma\mathbf{k}$, $\mathbf{b}=l\mathbf{i}+m\mathbf{j}+n\mathbf{k}$,

$\mathbf{a}'=\alpha'\mathbf{i}+\beta'\mathbf{j}+\gamma'\mathbf{k}$, $\mathbf{b}'=l'\mathbf{i}+m'\mathbf{j}+n'\mathbf{k}$.

These lines pass through the points $\mathbf{a}$, $\mathbf{a}'$ and are pararllel to the vectors $\mathbf{b}$, $\mathbf{b}'$ respectively.

If (1) and (2) are coplanar, their common plane will be parallel to the vectors $\mathbf{b}, \mathbf{b}'$.

Also, this common plane will contain the vector $(\mathbf{a}'-\mathbf{a})$.

$\therefore$ $\mathbf{a}'-\mathbf{a}$ is perpendicular to $\mathbf{b}\times\mathbf{b}'$.

or, $$(\mathbf{a}'-\mathbf{a}).(\mathbf{b}\times\mathbf{b}')=0,$$

or, $$[(\mathbf{a}'-\mathbf{a}), \mathbf{b}, \mathbf{b}']=0 \qquad \text{...(3)}$$

But $\mathbf{a}'-\mathbf{a}=(\alpha'-\alpha)\mathbf{i}+(\beta'-\beta)\mathbf{j}+(\gamma'-\gamma)\mathbf{k}$.

$\therefore$ (3) gives, $$\begin{vmatrix} \alpha'-\alpha & \beta'-\beta & \gamma'-\gamma \\ l & m & n \\ l' & m' & n' \end{vmatrix}=0,$$ which is the required condition.

Aliter.

Equation of **any** plane through the first line is

$$A(x-\alpha)+B(y-\beta)+C(z-\gamma)=0 \qquad \text{...(1)}$$

where $$Al+Bm+Cn=0 \qquad \text{...(2)}$$

If the second line lies in the plane (1), then this line is perpendicular to the normal to the plane (1), and its point $(\alpha', \beta', \gamma')$ lies on (1).

$\therefore$ $$Al'+Bm'+Cn'=0 \qquad \text{...(3)}$$

and $$A(\alpha'-\alpha)+B(\beta'-\beta)+C(\gamma'-\gamma)=0 \qquad \text{...(4)}$$

Eliminating A, B, C between (4), (2) and (3), we have

$$\begin{vmatrix} \alpha'-\alpha & \beta'-\beta & \gamma'-\gamma \\ l & m & n \\ l' & m' & n' \end{vmatrix}=0,$$ which is the required condition.

Cor. If the above given lines inteersect, to find the plane in which they lie.

From above, the vector equation of the plane containing the two given lines is

$$(\mathbf{r}-\mathbf{a}).\mathbf{b}\times\mathbf{b}'=0 \qquad \text{...(1)}$$

But $\mathbf{r}-\mathbf{a}=(x-\alpha)\mathbf{i}+(y-\beta)\mathbf{j}+(z-\gamma)\mathbf{k}$

$\therefore$ (1) becomes $[\{(x-\alpha)\mathbf{i}+(y-\beta)\mathbf{j}+(z-\gamma)\mathbf{k}\}, \mathbf{b}, \mathbf{b}']=0$,

or, $$\begin{vmatrix} x-\alpha & y-\beta & z-\gamma \\ l & m & n \\ l' & m' & n' \end{vmatrix}=0.$$

Aliter. Eliminating A, B, C between (1), (2) and (3), the equation of the plane containing the given lines is

$$\begin{vmatrix} \mathbf{x}-\alpha & \mathbf{y}-\beta & \mathbf{z}-\gamma \\ \mathbf{l} & \mathbf{m} & \mathbf{n} \\ \mathbf{l}' & \mathbf{m}' & \mathbf{n}' \end{vmatrix}=0.$$

Note. *We can also write down the condition of coplanarity of the given lines by writing the equation of a plane through the first line and parallel to the second line and satisfying it by any paint on the second line.*

4·12. To find the condition that the lines

$$\mathbf{a_1x+b_1y+c_1z+d_1=o}, \quad \mathbf{a_2x+b_2y+c_2z+d_2=o}$$

and

$$\mathbf{a_3x+b_3y+c_3z+d_3=o}, \quad \mathbf{a_4x+b_4y+c_4z+d_4=o}$$

may intersect (or are coplanar).

Any plane through the first line is

$$(a_1x+b_1y+c_1z+d_1)+\lambda(a_2x+b_2y+c_2z+d_2)=0,$$

or,
$$(a_1+\lambda a_2)x+(b_1+\lambda b_2)y+(c_1+\lambda c_2)z+(d_1+\lambda d_2)=0 \qquad ...(1)$$

Similarly, the equation of any plane through the second line is

$$(a_3+\mu a_4)x+(b_3+\mu b_4)y+(c_3+\mu c_4)z+(d_3+\mu d_4)=0 \qquad ...(2)$$

If the two lines intersect, then for some values of λ and μ, (1) and (2) will become identical.

$\therefore$ comparing coefficients of like terms, we have

$$\frac{a_1+\lambda a_2}{a_3+\mu a_4}=\frac{b_1+\lambda b_2}{b_3+\mu b_4}=\frac{c_1+\lambda c_2}{c_3+\mu c_4}=\frac{d_1+\lambda d_2}{d_3+\mu d_4}=k, \text{ say.}$$

$$\therefore \quad a_1+\lambda a_2+ka_3+\mu ka_4=0 \qquad ...(3)$$

$$b_1+\lambda b_2+kb_3+\mu kb_4=0 \qquad ...(4)$$

$$c_1+\lambda c_2+kc_3+\mu kc_4=0 \qquad ...(5)$$

$$d_1+\lambda d_2+kd_3+\mu kd_4=0 \qquad ...(6)$$

Eliminating λ, k, μk between (3), (4), (5) and (6), we have

$$\begin{vmatrix} \mathbf{a_1} & \mathbf{a_2} & \mathbf{a_3} & \mathbf{a_4} \\ \mathbf{b_1} & \mathbf{b_2} & \mathbf{b_3} & \mathbf{b_4} \\ \mathbf{c_1} & \mathbf{c_2} & \mathbf{c_3} & \mathbf{c_4} \\ \mathbf{d_1} & \mathbf{d_2} & \mathbf{d_3} & \mathbf{d_4} \end{vmatrix}=\mathbf{o},$$

which is the required condition.

Aliter.

The given lines intersect if the point of intersection of the first three planes lies on the fourth plane.

Let $(x_1, y_1\ z_1)$ be the point of intersection.

$$\therefore \qquad a_1x_1+b_1y_1+c_1z_1+d_1=0,$$
$$a_2x_1+b_2y_1+c_2z_1+d_2=0,$$
$$a_3x_1+b_3y_1+c_3z_1+d_3=0$$
and
$$a_4x_1+b_4y_1+c_4z_1+d_4=0.$$

Eliminating x_1, y_1, z_1 between these equations, we have

$$\begin{vmatrix} \mathbf{a}_1 & \mathbf{b}_1 & \mathbf{c}_1 & \mathbf{d}_1 \\ \mathbf{a}_2 & \mathbf{b}_2 & \mathbf{c}_2 & \mathbf{d}_2 \\ \mathbf{a}_3 & \mathbf{b}_3 & \mathbf{c}_3 & \mathbf{d}_3 \\ \mathbf{a}_4 & \mathbf{b}_4 & \mathbf{c}_4 & \mathbf{d}_4 \end{vmatrix} = \mathbf{0},$$ which is the required condition.

Note. *The condition of coplanarity is obtained by eliminating* x, y, z *between the given equations.*

4·13. Skew lines : Def.

Two straight lines which are neither intersecting nor parallel are called skew lines.

4·14. Shortest distance : Def.

If two straight lines are skew, then there is one and only one straight line which is perpendicular to both of them and this common perpendicular is called the shortest distance between the lines.

4·15. To find the length and the equations of the shortest distance between the given lines

$$\frac{\mathbf{x}-\alpha_1}{\mathbf{l}_1}=\frac{\mathbf{y}-\beta_1}{\mathbf{m}_1}=\frac{\mathbf{z}-\gamma_1}{\mathbf{n}_1}$$

and

$$\frac{\mathbf{x}-\alpha_2}{\mathbf{l}_2}=\frac{\mathbf{y}-\beta_2}{\mathbf{m}_2}=\frac{\mathbf{z}-\gamma_2}{\mathbf{n}_2}.$$

Let AB and CD be the given lines

$$\frac{x-\alpha_1}{l_1}=\frac{y-\beta_1}{m_1}=\frac{z-\gamma_1}{n_1}$$

and

$$\frac{x-\alpha_2}{l_2}=\frac{y-\beta_2}{m_2}=\frac{z-\gamma_2}{n_2}$$ respectively.

Let the corresponding vector equations of AB and CD be

$$\mathbf{r}=\mathbf{a}_1+t\mathbf{b}_1 \quad \text{and} \quad \mathbf{r}=\mathbf{a}_2+s\mathbf{b}_2$$

$\therefore \quad \mathbf{a}_1=\alpha_1\mathbf{i}+\beta_1\mathbf{j}+\gamma_1\mathbf{k}, \quad \mathbf{b}_1=l_1\mathbf{i}+m_1\mathbf{j}+n_1\mathbf{k}$

and $\quad \mathbf{a}_2=\alpha_2\mathbf{i}+\beta_2\mathbf{j}+\gamma_2\mathbf{k}, \quad \mathbf{b}_2=l_2\mathbf{i}+m_2\mathbf{j}+n_2\mathbf{k}.$

Let GH be the shortest distance between them.

$\because$ GH $\perp$ both $\mathbf{b}_1$ and $\mathbf{b}_2$,

$\therefore$ it is parallel to the vector $\mathbf{b}_1\times\mathbf{b}_2=\mathbf{n}$, say, whose modulus in n.

Let P $(\alpha_1, \beta_1, \gamma_1)$ and Q $(\alpha_2, \beta_2, \gamma_2)$ be the points on AB and CD respectively.

Now, the shortest distance

$$\begin{aligned} \text{GH} &= \text{the projection of PQ on GH} \\ &= \text{the projection of } \mathbf{a}_1-\mathbf{a}_2 \text{ on } \mathbf{n}. \\ &= \frac{(\mathbf{a}_1-\mathbf{a}_2)\cdot\mathbf{n}}{n}=\frac{(\mathbf{a}_1-\mathbf{a}_2)\cdot(\mathbf{b}_1\times\mathbf{b}_2)}{|\mathbf{b}_1\times\mathbf{b}_2|} \textbf{ (Vector form)}. \\ &= \frac{[(\mathbf{a}_1-\mathbf{a}_2), \mathbf{b}_1, \mathbf{b}_2]}{|\mathbf{b}_1\times\mathbf{b}_2|} \\ &= \begin{vmatrix} \alpha_1-\alpha_2 & \beta_1-\beta_2 & \gamma_1-\gamma_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} \div |\mathbf{b}_1\times\mathbf{b}_2| \end{aligned} \qquad ...(1)$$

Now, $\mathbf{b}_1\times\mathbf{b}_2=(l_1\mathbf{i}+m_1\mathbf{j}+n_1\mathbf{k})\times(l_2\mathbf{i}+m_2\mathbf{j}+n_2\mathbf{k})$

$\qquad =(m_1n_2-m_2n_1)\mathbf{i}+(n_1l_2-n_2l_1)\mathbf{j}+(l_1m_2-l_2m_1)\mathbf{k}$

$\therefore \quad |\mathbf{b}_1\times\mathbf{b}_2|=\sqrt{\Sigma(m_1n_2-m_2n_1)^2}$*.

$\therefore$ (1) gives

$$\text{GH}=\begin{vmatrix} \alpha_1-\alpha_2 & \beta_1-\beta_2 & \gamma_1-\gamma_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} \div \sqrt{\Sigma(m_1n_2-m_2n_1)^2},$$

which gives the length of the required S.D.

***We can also find $|\mathbf{b}_1\times\mathbf{b}_2|$ as follows :**

$$(\mathbf{b}_1\times\mathbf{b}_2)^2=b_1^2b_2^2\sin^2\theta,$$

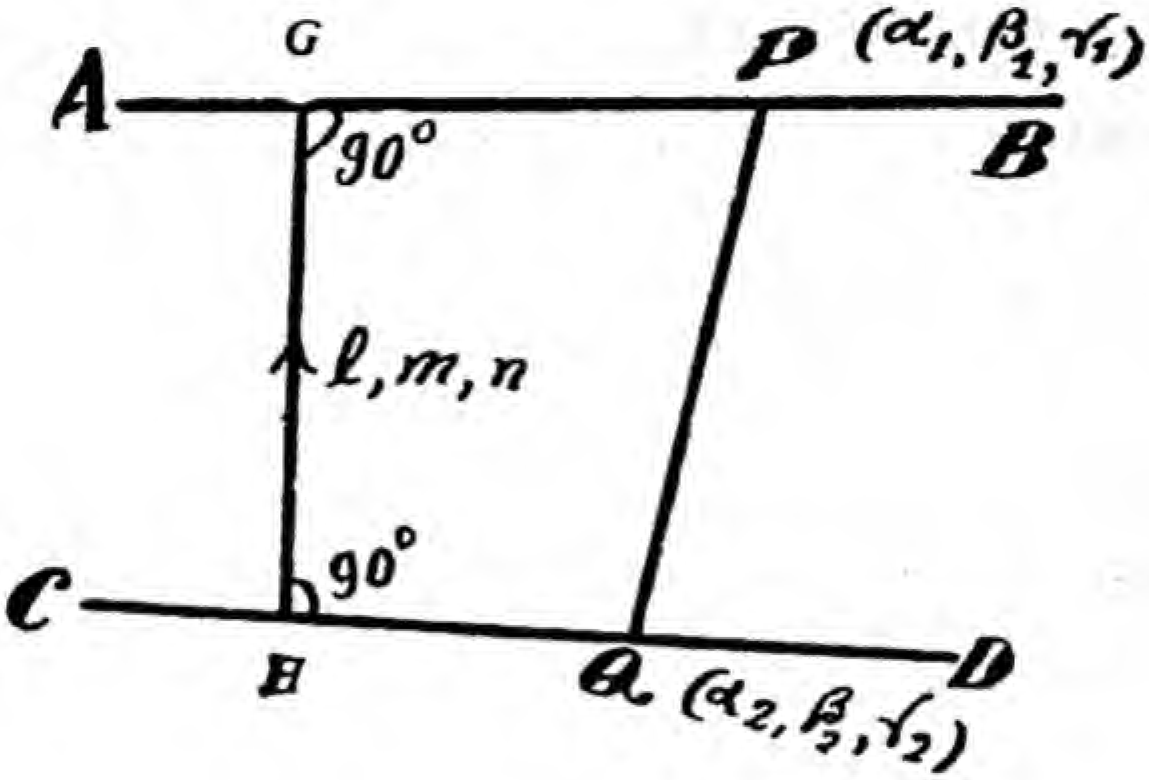
where θ is the angle between $\mathbf{b}_1$ and $\mathbf{b}_2$

$$\begin{aligned} &= b_1^2b_2^2(1-\cos^2\theta)=b_1^2b_2^2-(\mathbf{b}_1\cdot\mathbf{b}_2)^2 \\ &= (l_1^2+m_1^2+n_1^2)(l_2^2+m_2^2+n_2^2)-(l_1l_2+m_1m_2+n_1n_2)^2 \\ &= \Sigma(m_1n_2-m_2n_1)^2 \end{aligned}$$

$\therefore \quad |\mathbf{b}_1\times\mathbf{b}_2|=\sqrt{\Sigma(m_1n_2-m_2n_1)^2}.$

Aliter.

Let AB and CD be the two skew lines and GH be the shortest distance between them. Let the equations of AB and CD be respectively



$$\frac{x-\alpha_1}{l_1}=\frac{y-\beta_1}{m_1}=\frac{z-\gamma_1}{n_1} \qquad \ldots(1)$$

and

$$\frac{x-\alpha_2}{l_2}=\frac{y-\beta_2}{m_2}=\frac{z-\gamma_2}{n_2} \qquad \ldots(2)$$

Let P and Q be the points $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$ respectively.

Let l, m, n be the direction cosines of GH.

$\because$ GH is perpendicular to AB,

$\therefore$ $$ll_1+mm_1+nn_1=0 \qquad \ldots(3)$$

$\because$ GH is perpendicular to CD,

$\therefore$ $$ll_2+mm_2+nn_2=0 \qquad \ldots(4)$$

From (3) and (4), we have

$$\frac{l}{m_1n_2-m_2n_1}=\frac{m}{n_1l_2-n_2l_1}=\frac{n}{l_1m_2-l_2m_1}.$$

$\therefore$ direction cosines are projectional to $m_1n_2-m_2n_1$, $n_1l_2-n_2l_1$, $l_1m_2-l_2m_1$.

$\therefore$ actual direction cosines of GH are

$$\frac{m_1n_2-m_2n_1}{\sqrt{\Sigma(m_1n_2-m_2n_1)^2}}, \frac{n_1l_2-n_2l_1}{\sqrt{\Sigma(m_1n_2-m_2n_1)^2}}$$

and

$$\frac{l_1m_2-l_2m_1}{\sqrt{\Sigma(m_1n_2-m_2n_1)^2}}.$$

Length of shortest distance, GH :—

$$GH = \text{projection of } PQ \text{ on } GH$$
$$=(\alpha_2-\alpha_1)l+(\beta_2-\beta_1)m+(\gamma_2-\gamma_1)n$$
$$=\frac{(\alpha_2-\alpha_1)(\mathbf{m}_1\mathbf{n}_2-\mathbf{m}_2\mathbf{n}_1)+(\beta_2-\beta_1)(\mathbf{n}_1\mathbf{l}_2-\mathbf{n}_2\mathbf{l}_1)+(\gamma_2-\gamma_1)(\mathbf{l}_1\mathbf{m}_2-\mathbf{l}_2\mathbf{m}_1)}{\sqrt{\Sigma(\mathbf{m}_1\mathbf{n}_2-\mathbf{m}_2\mathbf{n}_1)^2}}$$

Equations of shortest distance :—

The equation of the line of shortest distance is the equation of the line of intersection of the planes through the given lines and the shortest distance.

The equation of the plane containing the line

$$\mathbf{r}=\mathbf{a}_1+t\mathbf{b}_1$$

and the shortest distance GH which is parallel to $\mathbf{b}_1\times\mathbf{b}_2$ and, therefore, perpendicular to $\mathbf{b}_1\times(\mathbf{b}_1\times\mathbf{b}_2)$ is

$$(\mathbf{r}-\mathbf{a}_1)\,.\,[\mathbf{b}_1\times(\mathbf{b}_1\times\mathbf{b}_2)]=0,$$

or, $$[(\mathbf{r}-\mathbf{a}_1),\,\mathbf{b}_1,\,(\mathbf{b}_1\times\mathbf{b}_2)]=0 \qquad ...(1)$$

Similarly, the equation of the plane containing

$$\mathbf{r}=\mathbf{a}_2+s\mathbf{b}_2 \text{ and GH is}$$
$$(\mathbf{r}-\mathbf{b}_2)\,.\,[\mathbf{b}_2\times(\mathbf{b}_1\times\mathbf{b}_2)]=0,$$

or, $$[(\mathbf{r}-\mathbf{a}_2),\,\mathbf{b}_2,\,(\mathbf{b}_1\times\mathbf{b}_2)]=0 \qquad ...(2)$$

The equation of the shortest distance is the equation of the line of intersection of (1) and (2), or of

$$\begin{vmatrix} x-\alpha_1 & y-\beta_1 & z-\gamma_1 \\ l_1 & m_1 & n_1 \\ m_1n_2-m_2n_1 & n_1l_2-n_2l_1 & l_1m_2-l_2m_1 \end{vmatrix}=0$$

and $$\begin{vmatrix} x-\alpha_2 & y-\beta_2 & z-\gamma_2 \\ l_2 & m_2 & n_2 \\ m_1n_2-m_2n_1 & n_1l_2-n_2l_1 & l_1m_2-l_2m_1 \end{vmatrix}=0.$$

Aliter. GH is the line of intersection of the planes PGH (through PB and GH) and QHG (through QD and GH.).

The equation of the plane through PB and GH is

$$\begin{vmatrix} x-\alpha_1 & y-\beta_1 & z-\gamma_1 \\ l_1 & m_1 & n_1 \\ m_1n_2-m_2n_1 & n_1l_2-n_2l_1 & l_1m_2-l_2m_1 \end{vmatrix}=0 \qquad ...(A)$$

The equation of the plane through QD and GH is

$$\begin{vmatrix} x-\alpha_2 & y-\beta_2 & z-\gamma_2 \\ l_2 & m_2 & n_2 \\ m_1n_2-m_2n_1 & n_1l_2-n_2l_1 & l_1m_2-l_2m_1 \end{vmatrix}=0 \qquad \text{...(B)}$$

$\therefore$ equations (A) and (B) determine the equations of the line of shortest distance.

Note 1. *The length of the shortest distance can also be written as*

$$\begin{vmatrix} \alpha_2-\alpha_1 & \beta_2-\beta_1 & \gamma_2-\gamma_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} \div \sqrt{\Sigma(m_1n_2-m_2n_1)^2}.$$

Note 2. *If the lines are coplanar, then the shortest distance between them is zero.*

EXAMPLES IV (E)

Type 1. Ex. 1. Show that the lines

$$\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z+2}{1}$$

and

$$\frac{x}{1}=\frac{y-7}{-3}=\frac{z+7}{2}$$

intersect. Find the coordinates of the point of intersection and the equation of the plane containing them. *(Punjab, 1957 ; Bombay, 1954)*

Sol. The given lines are

$$\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z+2}{1} \qquad \text{...(1)}$$

and

$$\frac{x}{1}=\frac{y-7}{-3}=\frac{z+7}{2} \qquad \text{...(2)}$$

Any point on (1) is $(-1-3R, 3+2R, -2+R)$.

If it lies on the line (2), then

$$\frac{-1-3R}{1}=\frac{3+2R-7}{-3}=\frac{-2+R+7}{2},$$

or,

$$\frac{-1-3R}{1}=\frac{2R-4}{-3}=\frac{R+5}{2}.$$

From first two members, we have

$$3+9R=2R-4, \text{ or, } 7R=-7, \therefore \quad R=-1.$$

This value of R satisfies the equation

$$\frac{2R-4}{-3}=\frac{R+5}{2}.$$

Hence lines (1) and (2) intersect.

The point of intersection is $(-1+3, 3-2, -2-1)$, or $(2, 1, -3)$.

The plane containing the lines (1) and (2) is

$$\begin{vmatrix} x+1 & y-3 & z+2 \\ -3 & 2 & 1 \\ 1 & -3 & 2 \end{vmatrix}=0,$$

or $(x+1)(4+3)-(y-3)(-6-1)+(z+2)(9-2)=0,$

or, $7x+7+7y-21+7z+14=0,$

or, $7x+7y+7z=0,$

or, $x+y+z=0.$

Aliter.

The equation of a plane containing (1) and parallel to (2) is

$$\begin{vmatrix} x+1 & y-3 & z+2 \\ -3 & 2 & 1 \\ 1 & -3 & 2 \end{vmatrix}=0,$$

or, $7(x+1)+7(y-3)+7(z+2)=0$, or, $x+y+z=0$ ...(A)

The point $(0, 7, -7)$ lying on (2) satisfies (A).

$\therefore$ the lines (1) and (2) are coplanar.

The plane containing the given lines is

$$x+y+z=0.$$

Any point on (1) is $(-1-3R, 3+2R, R-2)$.

Any point on (2) is $(R_1, 7-3R_1, 2R_1-7)$.

$\because$ (1) and (2) intersect,

$\therefore$ $-3R-1=R_1$, $2R+3=-3R_1+7$ and $R-2=2R_1-7$.

Solving first and second equations, we have

$$R=-1, \quad R_1=2.$$

These values satisfy the third equation.

The point of intersection is $(2, 1, -3)$.

Ex. 2. Prove that the lines

$$\frac{x-a}{a'}=\frac{y-b}{b'}=\frac{z-c}{c'}$$

and

$$\frac{x-a'}{a}=\frac{y-b'}{b}=\frac{z-c'}{c}$$

intersect, and find the coordinates of the point of intersection and the equation the plane in which they lie. *(Punjab B.Sc., 1958S)*

[**Ans.** $(a+a', b+b', c+c')$; $\begin{vmatrix} x & y & z \\ a' & b' & c' \\ a & b & c \end{vmatrix}=0.$]

Ex. 3. Prove that the lines

(i) $$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$$

and
$$\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$$

are coplanar. [*Punjab, 1961S* ; *Punjab (Pakistan), 1959* ; *Raj. Engg., 1963S* ; *A.M.I.E., May 1963*)]

(*ii*)
$$\frac{x-5}{4}=\frac{y-7}{4}=\frac{z+3}{-5}$$

and
$$\frac{x-8}{7}+\frac{y-4}{1}=\frac{z-5}{3}$$

are coplanar. (*Calcutta, 1961*)

Ex. 4. Prove that the lines
$$\frac{x-a+d}{\alpha-\delta}=\frac{y-a}{\alpha}=\frac{z-a-d}{\alpha+\delta}$$

and
$$\frac{x-b+c}{\beta-\gamma}=\frac{y-b}{\beta}=\frac{z-b-c}{\beta+\gamma}$$

are coplanar, and find the equation to the plane in which they lie.

[*Punjab (Pakistan), 1955, Allahabad, 1956*)

[**Ans.** $2y-x-z=0$.]

Ex. 5. Prove that the lines
$$\frac{x+1}{3}=\frac{y+3}{5}=\frac{z+5}{7}$$

and
$$\frac{x-2}{1}=\frac{y-4}{3}=\frac{z-6}{5}$$

intersect and find the equation of the plane in which they lie.

(*Punjab B.Sc., 1959S*)

[**Ans.** $x-2y+z=0$.]

Type II. Ex. 1. Prove that the lines $x=ay+b=cz+d$ and $x=\alpha y+\beta=\gamma z+\delta$ are coplanar if $(\gamma-c)(a\beta-b\alpha)-(\alpha-a)(c\delta-d\gamma)=0$.

(*Bombay, 1953*)

Sol. The given lines are
$$\frac{x}{ac}=\frac{y+\frac{b}{a}}{c}=\frac{z+\frac{d}{c}}{a} \qquad \text{...(1)}$$

and
$$\frac{x}{\alpha\gamma}=\frac{y+\frac{\beta}{\alpha}}{\gamma}=\frac{z+\frac{\delta}{\gamma}}{\alpha} \qquad \text{...(2)}$$

Equation of the plane through (1) and parallel to (2) is
$$\begin{vmatrix} x & y+\frac{b}{a} & z+\frac{d}{c} \\ ac & c & a \\ \alpha\gamma & \gamma & \alpha \end{vmatrix}=0 \qquad \text{...(3)}$$

The lines (1) and (2) are coplanar if the point

$\left(0, -\frac{\beta}{\alpha}, -\frac{\delta}{\gamma}\right)$ of (2) lies on (3). This requires that

$$\begin{vmatrix} 0 & \frac{b}{a}-\frac{\beta}{\alpha} & \frac{d}{c}-\frac{\delta}{\gamma} \\ ac & c & a \\ \alpha\gamma & \gamma & \alpha \end{vmatrix}=0,$$

or, $$\left(-\frac{b}{a}-\frac{\beta}{\alpha}\right)\left(ac\alpha-\alpha\gamma a\right)+\left(\frac{d}{c}-\frac{\delta}{\gamma}\right)\left(ac\gamma-c\alpha\gamma\right)=0.$$

or, $$(a-\alpha)(d\gamma-c\delta)-(c-\gamma)(b\alpha-a\beta)=0,$$

or, $$(\gamma-c)(a\beta-b\alpha)-(\alpha-a)(c\delta-d\gamma)=0,$$

Aliter.

The required condition is obtained by eliminating x, y, z from the given equations.

Now $x=ay+b$, $x=\alpha y+\beta$ gives

$$x(\alpha-a)=b\alpha-a\beta \qquad \ldots(A)$$

Again, $$x=cz+d, \quad x=\gamma z+\delta \text{ gives}$$

$\therefore$ $$x(\gamma-c)=d\gamma-c\delta \qquad \ldots(B)$$

From (A) and (B), on division, we have

$$\frac{\alpha-a}{\gamma-c}=\frac{b\alpha-a\beta}{d-c\delta} \quad \text{or, } (\gamma-c)(a\beta-b\alpha)-(\alpha-a)(c\delta-d\gamma)=0.$$

Ex. 2. Show that the lines $x=az+b, y=cz+d$ and $x=\alpha z+\beta$, $y=\gamma z+\delta$ intersect if $(\beta-b)(\gamma-c)=(\delta-d)(\alpha-a)$.

Ex. 3. Show that the lines $x-1=2y-4=3z$, $3x-5=4y-9=3z$ meet in a point and the equation of the plane in which they lie is $3x-8y+3z+13=0$.

Type III. Ex. 1. The equations of the four planes are $x+2y-z-3=0$, $3x-y+2z-1=0$, $2x-2y+3z-2=0$ and $x-y+z+1=0$. Show that the line of intersection of the first two planes is coplanar with the line of intersection of the latter two and find the equation of the plane containing the two lines. *(A.M.I.E., May, 1958)*

Sol. Solving first three planes, we have]

$$x=-1, y=4, z=4.$$

Now, the point $(-1, 4, 4)$ satisfies the fourth equation of the plane.

$\therefore$ the two lines are coplanar.

Aliter. The equation of the first line in the symmetrical form is

$$\frac{x}{-3}=\frac{y-7/3}{5}=\frac{z-5/3}{7}* \qquad \ldots(1)$$

*The direction ratios of the first line are give by

$$x+2y-z=0 \text{ and } 3x-y+2z=0.$$

$\therefore$ $$\frac{x}{-3}=\frac{y}{5}=\frac{z}{7}.$$

$\therefore$ direction ratios are $-3, 5, 7$.

Putting $x=0$ in the first two equations, we have

$$2y-z-3=0 \text{ and } -y+2z-1=0 \quad \therefore\ y=7/3,\ z=5/3.$$

$\therefore$ any point on the line is $(0, 7/3, 5/3)$.

Any plane through the line of intersection of the last two planes is

$$2x-2y+3z-2+\lambda(x-y+z+1)=0$$

It is parallel to (1) if $-3(2+\lambda)+5(-2-\lambda)+7(3+\lambda)=0$, or, $\lambda=5$.

$\therefore$ the plane through the second line parallel to the first line is

$$2x-2y+3z-2+5(x-y+z+1)=0, \text{ or, } 7x-7y+8z+3=0 \quad \ldots(2)$$

Now, any point $(0, 7/3, 5/3)$ lying on (1) satisfies (2).

$\therefore$ the lines are coplanar.

The equation of the plane containing them is given by (2).

Ex. 2. Find the condition that the lines $\dfrac{x-x_1}{l}=\dfrac{y-y_1}{m}=\dfrac{z-z_1}{n}$ and $ax+by+cz+d=0=a'x+b'y+c'z+d'$ are coplanar. *(Raj., 1959)*

[**Ans.** $(ax_1+by_1+cz_1+d)(a'l+b'm+c'n)$
$=(a'x_1+b'y_1+c'z_1+d')(al+bm+cn)$].

Ex. 3. Show that the lines $2x-y+3z+3=0$, $x+10y-21=0$ and $2x-y=0=7x+z-6$ intersect. Find the coordinates of the point of intersection and also the equation of the plane containing these lines. *(Bombay, 1956)*

[**Ans.** $(1, 2, -1)$, $x+3y+z-6=0$].

Ex. 4. Prove that the lines $\dfrac{x-1}{2}=\dfrac{y-2}{3}=\dfrac{z-3}{4}$ and $4x-3y+1=0=5x-3z+2$ are coplanar. *(Calcutta, 1963)*

Type IV. Ex. 1. Two straight lines $\dfrac{x-\alpha}{l}=\dfrac{y-\beta}{m}=\dfrac{z-\gamma}{n}$; $\dfrac{x-\alpha'}{l'}=\dfrac{y-\beta'}{m'}=\dfrac{z-\gamma'}{n'}$ are cut by a third line whose direction cosines are λ, μ, ν. Show that the length intercepted on the third line is given by

$$\begin{vmatrix} \alpha-\alpha' & \beta-\beta' & \gamma-\gamma' \\ l & m & n \\ l' & m' & n' \end{vmatrix} \div \begin{vmatrix} l & m & n \\ l' & m' & n' \\ \lambda & \mu & \nu \end{vmatrix}.$$

Hence deduce the length of the shortest distance.

(Allahabad, 1952 ; Agra, 1951 ; Raj., 1960 ; Vikram, 1961 ; Bihar, 1961)

Sol. Any point P on the first line is $(\alpha+lr, \beta+mr, \gamma+nr)$

Let us suppose that the third line whose direction consines are λ, μ, ν meet the first line in this point $(\alpha+lr, \beta+mr, \gamma+nr)$.

$\therefore$ the equation of the third line are

$$\frac{x-(\alpha+lr)}{\lambda}=\frac{y-(\beta+mr)}{\mu}=\frac{z-(\gamma+nr)}{\nu}.$$

Let this line meet the second line in P_1 whose distance from P is d, say.

$\therefore$ coordinates of P_1 are $[\lambda d+(\alpha+lr), \mu d+(\beta+mr), \nu d+(\gamma+nr)]$.

Again, if the third line meets the second line in the point $(\alpha'+l'r', \beta'+m'r', \gamma'+n'r')$, then

$$\lambda d+\alpha+lr = \alpha'+l'r'$$

$$\mu d+\beta+mr = \beta'+m'r'$$
$$\nu d+\gamma+nr = \gamma'+n'r',$$

or,
$$\lambda d+\alpha-\alpha'+lr-l'r' = 0,$$
$$\mu d+\beta-\beta'+mr-m'r' = 0,$$
$$\nu d+\gamma-\gamma'+nr-n'r' = 0.$$

Eliminating r and r' we have

$$\begin{vmatrix} \lambda d+\alpha-\alpha' & l & l' \\ \mu d+\beta-\beta' & m & m' \\ \nu d+\gamma-\gamma' & n & n' \end{vmatrix} = 0, \text{ or, } d\begin{vmatrix} \lambda & l & l' \\ \mu & m & m' \\ \nu & n & n' \end{vmatrix} + \begin{vmatrix} \alpha-\alpha' & l & l' \\ \beta-\beta' & m & m' \\ \gamma-\gamma' & n & n' \end{vmatrix} = 0,$$

or,
$$d\begin{vmatrix} \lambda & \mu & \nu \\ l & m & n \\ l' & m' & n' \end{vmatrix} = -\begin{vmatrix} \alpha-\alpha' & \beta-\beta' & \gamma-\gamma' \\ l & m & n \\ l' & m' & n' \end{vmatrix}$$

$$\therefore \; d = \begin{vmatrix} \alpha-\alpha' & \beta-\beta' & \gamma-\gamma' \\ l & m & n \\ l' & m' & n' \end{vmatrix} \div \begin{vmatrix} l & m & n \\ l' & m' & n' \\ \lambda & \mu & \nu \end{vmatrix} \quad \ldots\text{(A) (in magnitude)}$$

If d stands for the shortest distance, then the third line is perpendicular to the given lines.

$\therefore \; \lambda l+\mu m+\nu n=0$ and $\lambda l'+\mu m'+\nu n'=0$

$$\therefore \quad \frac{\lambda}{mn'-m'n} = \frac{\mu}{nl'-n'l} = \frac{\nu}{lm'-l'm} = \frac{\sqrt{\lambda^2+\mu^2+\nu^2}}{\sqrt{\Sigma(mn'-m'n)^2}}$$
$$= \frac{1}{\sqrt{\Sigma(mn'-m'n)^2}}.$$

Now,
$$\begin{vmatrix} l & m & n \\ l' & m' & n' \\ \lambda & \mu & \nu \end{vmatrix} = \begin{vmatrix} \lambda & \mu & \nu \\ l & m & n \\ l' & m' & n' \end{vmatrix} = \lambda(mn'-m'n)+\mu(nl'-n'l)+\nu(lm'-l'm)$$

$$= \frac{(mn'-m'n)^2+(nl'-n'l)^2+(lm'-l'm)^2}{\sqrt{\Sigma(mn'-m'n)^2}} = \sqrt{\Sigma(mn'-m'n)^2}$$

$\therefore$ (A) becomes, $d=$ shortest distance

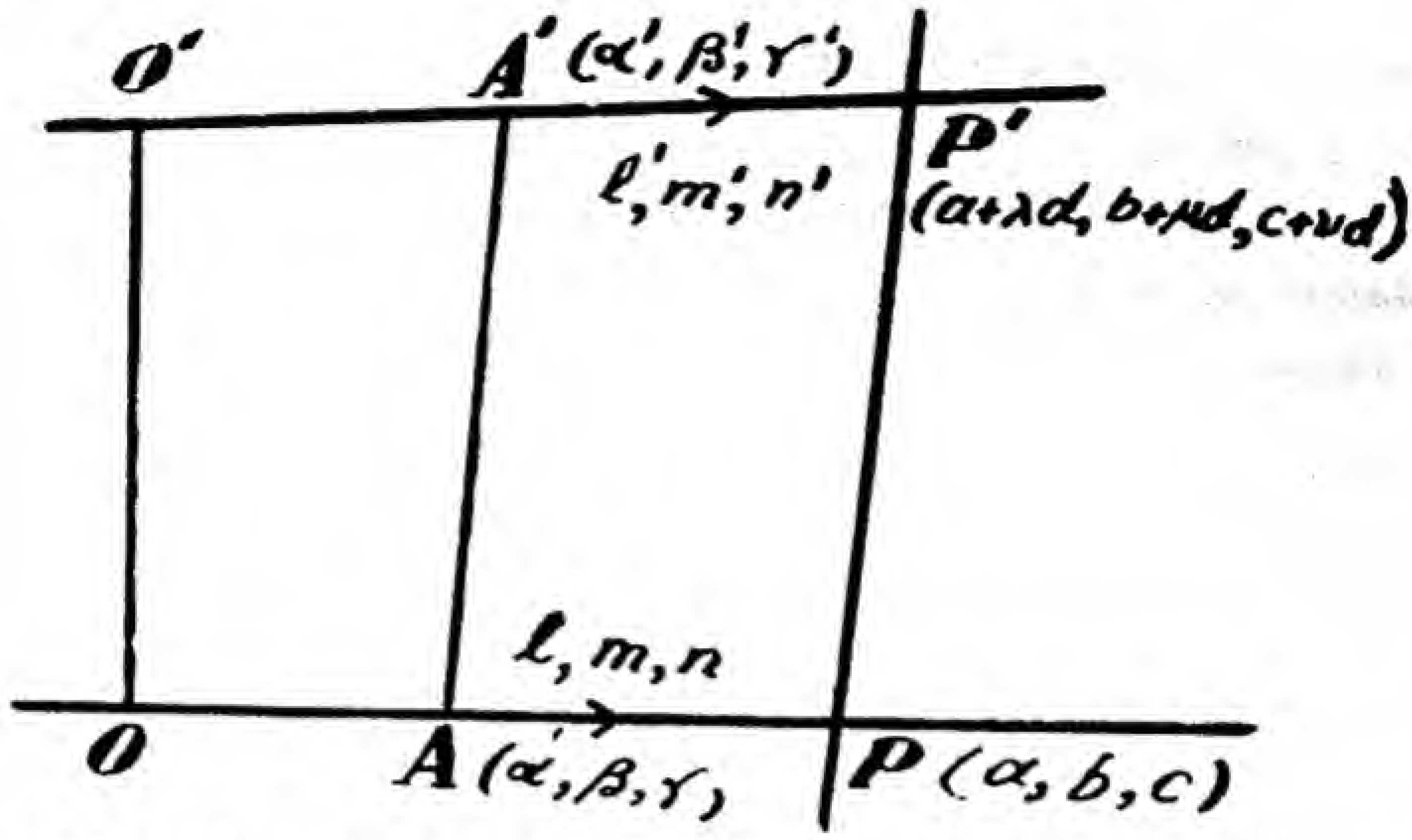
$$= \begin{vmatrix} \alpha-\alpha' & \beta-\beta' & \gamma-\gamma' \\ l & m & n \\ l' & m' & n' \end{vmatrix} \div \sqrt{\Sigma(mn'-m'n)^2}.$$

Aliter.

Let the third line meet the given lines at P and P′ respectively, where PP′$=d$, say. Let P be (a, b, c).

$\therefore$ co-ordinates of P′ are $(a+\lambda d, b+\mu d, c+\nu d)$,

Let A and A′ be the points (α, β, γ) and $(\alpha', \beta', \gamma')$ respectively. Join AA′, PP′.



Let OO′ be the shortest distance between the given lines. Projecting AA′ on OO′, we have

shortest distance$=\Sigma(\alpha-\alpha')(mn'-m'n)\div\sqrt{\Sigma(mn'-m'n)^2}$,
where $mn'-m'n$, $nl'-n'l$, $lm'-l'm$ are the direction ratios of OO′,
or, the shortest distance$=\Sigma(\alpha-\alpha')(mn'-m'n)\div\sin\theta$ ...(1)
where θ is the angle between the given lines.

Again, projecting PP′ on OO′, the shortest distance

$$=\frac{(a+\lambda d-a)(mn'-m'n)+(b+\mu d-b)(nl'-n'l)+(c+\nu d-c)(lm'-l'm)}{\sin\theta} \quad ...(2)$$

$\because$ (1) and (2) are equal,

$\therefore d[\Sigma\lambda(mn'-m'n)]=\Sigma(\alpha-\alpha')(mn'-m'n)$.

$$\therefore d=\begin{vmatrix}\alpha-\alpha' & \beta-\beta' & \gamma-\gamma' \\ l & m & n \\ l' & m' & n'\end{vmatrix}\div\begin{vmatrix} l & m & n \\ l' & m' & n' \\ \lambda & \mu & \nu\end{vmatrix} \quad ...(A)$$

Now, PP′ $(=d)$ will be the shortest distance if

$$\lambda=\frac{m'n-m'n}{\sin\theta},\ \mu=\frac{nl'-n'l}{\sin\theta},\ \nu=\frac{lm'-l'm}{\sin\theta}.$$

$\therefore$ PP′=shortest distance

$$=\frac{\Sigma(\alpha-\alpha')(mn'-m'n)}{\dfrac{\Sigma(mn'-m'n)^2}{\sin\theta}},\text{ from (A)}$$

$$=\Sigma(\alpha-\alpha')(mn'-m'n)\div\sqrt{\Sigma(mn'-m'n)^2}.$$

Ex. 2. A line with direction cosines proportional to 2, 7, −5 is drawn to intersect the lines,

$$\frac{x-5}{3}=\frac{y-7}{-1}=\frac{z+2}{1};\ \frac{x+3}{-3}=\frac{y-3}{2}=\frac{z-6}{4}.$$

Find the co-ordinates of the points of intersection and the length intercepted on it.
[*Punjab (Pakistan), 1955 S ; Raj., 1956*]

[**Ans.** (2, 8, −3), (0, 1, 2); $\sqrt{78}$.]

Type V. Ex. 1. Find the shortest distance between the lines $\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}$; $\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}$.

(A.M.I.E., Nov., 1959 ; Kashmir, 1958 ; Vikram Engg., 1959 ; Poona, 1953)

Find also the equation and the points in which it meets the given lines.

(Agra, 1953 ; Punjab T.D.C., 1964 ; Banaras, 1961)

Sol. Let the line of shortest distance meet the given lines in P_1 and P_2 respectively. Then the co-ordinates of P_1 and P_2 may be written as

$(3+3R, 8-R, 3+R)$ and $(-3-3R', -7+2R', 6+4R')$.

$\therefore$ direction ratios of P_1P_2 are $6+3R+3R'$, $15-R-2R'$, $-3+R-4R'$.

$\because$ P_1P_2 is at right angles to the given lines,

$\therefore \quad 3(6+3R+3R')-(15-R-2R')+(-3+R-4R')=0$

and $\quad -3(6+3R+3R')+2(15-R-2R')+4(-3+R-4R')=0,$

or, $\quad 11R+7R'=0$ and $-7R-29R'=0,$

$\therefore \quad R=R'=0.$

$\therefore$ P_1 and P_2 are the points $(3, 8, 3)$, $(-3, -7, 6)$.

Also, $P_1P_2=\sqrt{36+225+9}=3\sqrt{30}.$

Further, the equations of P_1P_2 are

$$\frac{x-3}{2}=\frac{y-8}{5}=\frac{z-3}{-1}.$$

Aliter. *We can also find the length of the shortest distance as follows :*

The shortest distance is the perpendicular distance of any point on the first line from the plane drawn through the second line parallel to the first line.

The equation of the plane through the second line is

$$A(x+3)+B(y+7)+C(z-6)=0 \qquad \ldots(1)$$

where
$$-3A+2B+4C=0 \qquad \ldots(2)$$

$\because$ (1) is parallel to the first line,

$$\therefore \quad 3A-B+C=0 \qquad \ldots(3)$$

From (2) and (3), we have

$$\frac{A}{2}=\frac{B}{5}=\frac{C}{-1}.$$

$\therefore$ (1) becomes $2(x+3)+5(y+7)-(z-6)=0,$

or, $$2x+5y-z+47=0 \qquad \ldots(4)$$

Any point on the first line is $(3+3R, 8-R, 3+R)$.

$\therefore$ required shortest distance

=perpendicular distance of this point from (4)

$$=\frac{2(3+3r)+5(8-r)-(3+r)+47}{\sqrt{4+25+1}}=\frac{90}{\sqrt{30}}=3\sqrt{30}.$$

Aliter.

Let the given lines pass through the points A_1

$\mathbf{a}_1=3\mathbf{i}+8\mathbf{j}+3\mathbf{k}$ and A_2, $\mathbf{a}_2=-3\mathbf{i}-7\mathbf{j}+6\mathbf{k}$,

and are parallel to the vectors

$$\mathbf{b}_1=3\mathbf{i}-\mathbf{j}+\mathbf{k} \text{ and } \mathbf{b}_2=-3\mathbf{i}+2\mathbf{j}+4\mathbf{k}$$

respectively.

Let P_1P_2 be the shortest distance.

Now $\mathbf{b}_1\times\mathbf{b}_2=(3\mathbf{i}-\mathbf{j}+\mathbf{k})\times(-3\mathbf{i}+2\mathbf{j}+4\mathbf{k})$

$=(6\mathbf{k}-12\mathbf{j}-3\mathbf{k}-4\mathbf{i}-3\mathbf{j}-2\mathbf{i})$

$=(-6\mathbf{i}-15\mathbf{j}+3\mathbf{k})$

$\therefore$ P_1P_2=projection of A_2A_1 on P_1P_2

$=(\overrightarrow{A_2A_1})$. (unit vector along P_1P_2)

$$=(-6\mathbf{i}-15\mathbf{j}+3\mathbf{k}).\frac{(-6\mathbf{i}-15\mathbf{j}+3\mathbf{k})}{\sqrt{270}}$$

$$=\frac{(36+225+9)}{\sqrt{270}}=\sqrt{270}=3\sqrt{30}.$$

The length and the equations of the shortest distance can also be found as under:

Let A and A′ be the given points (3, 8, 3), (−3, −7, 6) on the given lines respectively.

Let λ, μ, ν be the direction cosines of the shortest distance.

$\because$ The shortest distance is perpendicular to the given lines,

$\therefore$ $$3\lambda-\mu+\nu=0$$

and $$-3\lambda+2\mu+4\nu=0$$

Solving, we have $\dfrac{\lambda}{-2}=\dfrac{\mu}{-5}=\dfrac{\nu}{1}$.

Now, S.D.=projection of AA′ on the line of shortest distance OO′

$$=(-3-3)\frac{(-2)}{\sqrt{30}}+(-8-7)\frac{(-5)}{\sqrt{30}}+(6-3).\frac{1}{\sqrt{30}}$$

$$=\frac{12}{\sqrt{30}}+\frac{75}{\sqrt{30}}+\frac{3}{\sqrt{30}}=\frac{90}{\sqrt{30}}=3\sqrt{30}.$$

Now, the equation of the shortest distance is given by the equations of two planes AOO′ and A′OO′ as

$$\begin{vmatrix} x-3 & y-8 & z-3 \\ 3 & -1 & 1 \\ -2 & -5 & 1 \end{vmatrix}=0 \text{ and } \begin{vmatrix} x+3 & y+7 & z-6 \\ -3 & 2 & 4 \\ -2 & -5 & 1 \end{vmatrix}=0$$

or, $$(x-3)(-1+5)-(y-8)(3+2)+(z-3)(-15-2)=0$$

and $$(x+3)(2+20)-(y+7)(-3+8)+(z-6)(15+4)=0,$$

or, $$4x-5y-17z+79=0$$

and $$22x-5y+19z-83=0.$$

Ex. 2. Find the length and equations of the shortest distance between the following lines :

(*i*) $\dfrac{x+3}{-4}=\dfrac{y-6}{6}=\dfrac{z}{2}$; $\dfrac{x+2}{-4}=\dfrac{y}{1}=\dfrac{z-7}{1}$.

(*Marathwada T. D. C, 1961*)

(*ii*) $\dfrac{x-3}{1}=\dfrac{y-5}{-2}=\dfrac{z-7}{1}$; $\dfrac{x+1}{7}=\dfrac{y+1}{-6}=\dfrac{z+1}{1}$.

(*Punjab, 1959 ; Raj. Engg., 1963 ; A.M.I.E., May, 1963*)

Also find the points where the line of shortest distance intersects the given lines. (*Punjab, 1959 ; Agra, 1962*)

(*iii*) $\dfrac{x-1}{2}=\dfrac{y-2}{3}=\dfrac{z-3}{4}$; $\dfrac{x-2}{3}=\dfrac{y-4}{4}=\dfrac{z-5}{5}$.

(*Punjab, 1958 S ; Karnatak, 1962 ; Sagar, 1964*)

(*iv*) $\dfrac{x-8}{3}=\dfrac{y+9}{-16}=\dfrac{z-10}{7}$; $\dfrac{x-15}{3}=\dfrac{y-29}{8}=\dfrac{z-5}{-5}$. (*A.M.I.E., 1957*)

(*v*) $\dfrac{x}{2}=\dfrac{y}{-3}=\dfrac{z}{1}$ and $\dfrac{x-2}{3}=\dfrac{y-1}{-5}=\dfrac{z+2}{2}$. (*A.M.I.E., 1961*)

[**Ans.** (*i*) $\dfrac{14}{\sqrt{3}}$, $16x+11y-z=18$, $2x+7y+z=3$;

(*ii*) $2\sqrt{29}$; $\dfrac{x-1}{2}=\dfrac{y-2}{3}=\dfrac{z-3}{4}$; $(3, 5, 7)$, $(-1,-1,-1)$;

(*iii*) $\dfrac{1}{\sqrt{6}}$; $11x+2y-7z+6=0$, $7x+y-5z+7=0$;

(*iv*) 14, $117x+4y-41z=490$, $9x-4y-z=14$;

(*v*) $\dfrac{1}{\sqrt{3}}$, $4x+y-5z=0$, $7x+y-8z=31$.]

Type VI. Ex. (*i*) **Show that the shortest distance between the line $\mathbf{ax+by+cz+d=0=a'x+b'y+c'z+d'=0}$ and the z-axis meets the z-axis at a point whose distance from the origin is**

$$[(\mathbf{db'-d'b})(\mathbf{bc'-b'c})+(\mathbf{ca'-c'a})(\mathbf{ad'-a'd})]$$
$$\div[(\mathbf{bc'-b'c})^2+(\mathbf{ca'-c'a})^2].$$

(*Delhi Hons., 1958 ; Raj., 1952 ; Agra, 1961*)

(*ii*) **Find the S.D. between the axis of z and the line $\mathbf{ax+by+cz+d=0}$, $\mathbf{a'x+b'y+c'z+d'=0}$.**

(*Delhi Hons., 1958 ; Punjab B.Sc , 1959S ; Bombay, 1962*)

Sol. (*i*) The equation of the first line in the symmetrical form is

$$\frac{x-\dfrac{bd'-b'd}{ab'-a'b}}{bc'-b'c}=\frac{y-\dfrac{da'-d'a}{ab'-a'b}}{ca'-c'a}=\frac{z}{ab'-a'b}=\text{R, say.}$$

Any point on this line is

$$\text{P}\left[\frac{bd'-b'd}{ab'-a'b}+\text{R}(bc'-b'c),\ \frac{da'-d'a}{ab'-a'b}+\text{R}(ca'-c'a),\ \text{R}(ab'-a'b).\right]$$

Equations of z-axis are $\dfrac{x}{0}=\dfrac{y}{0}=\dfrac{z}{1}=r.$

$\therefore$ the coordinates of the point Q on the z-axis at a distance r from O is $(0, 0, r)$.

Let PQ be the line of shortest distance. The direction ratios of PQ are

$$\frac{bd'-b'd}{ab'-a'b}+\text{R}\,(bc'-b'c),\ \frac{da'-d'a}{ab'-a'b}+\text{R}(ca'-c'a),$$
$$\text{R}(ab'-a'b)-r.$$

$\because$ PQ is perpendicular to both the lines,

$$\therefore\quad \left[\frac{bd'-b'd}{ab'-a'b}+\text{R}(bc'-b'c)\right](bc'-b'c)$$
$$+\left[\frac{da'-d'a}{ab'-a'b}+\text{R}(ca'-c'a)\right](ca'-c'a)$$
$$+[\text{R}(ab'-a'b)-r]\,(ab'-a'b)=0 \qquad \ldots(1)$$

and
$$[\text{R}(ab'-a'b)-r]=0 \qquad \ldots(2)$$

Eliminating R between (1) and (2), we have

$$\frac{(bd'-b'd)+r(bc'-b'c)}{ab'-a'b}\cdot(bc'-b'c)+\frac{(da'-d'a)+r(ca'-c'a)}{ab'-a'b}(ca'-c'a)=0$$

or,
$$r=(bc'-b'c)(db'-bd')+(ca'-c'a)(ad'-a'd)$$
$$\div[(bc'-b'c)^2+(ca'-c'a)^2]$$

(*ii*) Equation of the plane through the second line is

$$(ax+by+cz+d)+\lambda(a'x+b'y+c'z+d')=0$$

or,
$$(a+\lambda a')x+(b+\lambda b')y+(c+\lambda c')z+(d+\lambda d')=0 \qquad \ldots(1)$$

Equations of z-axis are

$$\frac{x}{0}=\frac{y}{0}=\frac{z}{1} \qquad \ldots(2)$$

If (1) is parallel to (2), then

$$(a+\lambda a')(0)+(b+\lambda b')(0)+(c+\lambda c')=0,$$

or,
$$\lambda=-\frac{c}{c'}.$$

$\therefore$ (1) becomes $(ax+by+cz+d)-\dfrac{c}{c'}(a'x+b'y+c'z+d')=0,$

or,
$$c'(ax+by+cz+d)-c(a'x+b'y+c'z+d')=0,$$

or,
$$x(ca'-c'a)+y(cb'-c'b)+(cd'-c'd)=0 \qquad \ldots(3)$$

Any point on the line (2) is $(0, 0, r)$

$\therefore$ Shortest distance required = length of the perpendicular from $(0, 0, r)$ on (3)

$$=(cd'-c'd)\div\sqrt{(ca'-c'a)^2+(cb'-c'b)^2}.$$

Ex. 2. Find the magnitude of the shortest distance between the lines

$$\frac{x-3}{1}=\frac{y-4}{-2}=\frac{z+2}{-1}$$

and the line $3x-y-10=0,\ 2x-z-4=0.$ (*A.M.I.E., Nov. 1960*)

[**Ans.** $\sqrt{35}$.]

Ex. 3. Find the shortest distance between the lines

$$(i)\ \frac{x-7}{2}=\frac{y+4}{3}=\frac{z-2}{1}$$

and
$$2x+5y-8z-52=0=3x-3y+2z+27.$$

(*Bombay, 1955 ; Poona, 1952*)

[**Ans.** $6\sqrt{5}$.]

$$(ii)\ \frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}$$

and $$2x+y+z+7=0=16x+9y+9z+36.$$

(*Bombay, 1950* ; *Gujarat, 1953*)

[**Ans.** $9/\sqrt{2}$.]

Type VII. Ex. 1. If the axes are rectangular, the shortest distance between the lines $y=az+b$, $z=\alpha x+\beta$; $y=a'z+b'$, $z=\alpha' x+\beta'$ is

$$\frac{(\alpha-\alpha')(b-b')-(\alpha'\beta-\alpha\beta')(a-a')}{[\alpha^2\alpha'^2(a-a')^2+(\alpha-\alpha')^2+(a\alpha-a'\alpha')^2]^{\frac{1}{2}}}.$$

(*Punjab, 1952 S* ; *Raj., 1962*)

Sol. The equations of the given lines are

$$\frac{x+\beta/\alpha}{1}=\frac{y-b}{\alpha a}=\frac{z}{\alpha} \qquad ...(1)$$

and $$\frac{x+\beta'/\alpha'}{1}=\frac{y-b'}{a'\alpha'}=\frac{z}{\alpha'} \qquad ...(2)$$

Let λ, μ, ν be the direction cosines of the line of shortest distance.

$\because$ it is perpendicular to (1) and (2),

$$\therefore \quad \lambda+a\alpha\mu+\alpha\nu=0 \qquad ...(3)$$

and $$\lambda+a'\alpha'\mu+\alpha'\nu=0 \qquad ...(4)$$

From (3) and (4), we have

$$\frac{\lambda}{\alpha\alpha'(a-a')}=\frac{\mu}{\alpha-\alpha'}=\frac{\nu}{a'\alpha'-a\alpha}.$$

$\therefore$ Shortest distance

= projection of the line joining the points $\left(-\frac{\beta}{\alpha}, b, 0\right)$ and $\left(-\frac{\beta'}{\alpha'}, b', 0\right)$ on the line with direction cosines λ, μ, ν

$$=\left(-\frac{\beta}{\alpha}+\frac{\beta'}{\alpha'}\right)\frac{\alpha\alpha'(a-a')}{D}+(b-b')\frac{(\alpha-\alpha')}{D},$$

where $D^2\equiv\alpha^2\alpha'^2(a-a')^2+(\alpha-\alpha')^2+(a'\alpha'-a\alpha)^2$

or shortest distance $=\dfrac{(\alpha-\alpha')(b-b')-(\alpha'\beta-\alpha\beta')(a-a')}{\sqrt{\alpha^2\alpha'^2(a-a')^2+(\alpha-\alpha')^2+(a'\alpha'-a\alpha)^2}}.$

Ex. 2. Find the equation and magnitude of the shortest distance between the lines $x+a=2y=-12z$, $x=y+2a=6z-6a$.

(*Karnatak Engg., 1961*)

[**Ans.** $2a$.]

Ex. 3. Find the shortest distance between the lines

$$x=2z+3, \quad y=3z+4 ; \quad x=4z+5, \quad y=5z+6.$$

What conclusions do you draw from your answer ? (*Poona, 1955*)

[**Ans.** zero ; the lines are coplanar.]

Type VII. Ex. 1. Show that the shortest distance between any two opposite edges of the tetrahedron formed by the planes $y+z=0$, $z+x=0$, $x+y=0$, $x+y+z=a$ is $\frac{2a}{\sqrt{6}}$.

(*Kashmir, 1953* ; *Jodhpur, 1964* ; *Raj., 1960* ; *Bombay, 1956* ; *Aligarh, 1962* ; *Punjab T.D.C., 1965 S*)

Show further that the three lines of shortest distance intersect at the point $x=y=z=-a$. [*Delhi Hons., 1960 ; Punjab (Pakistan), 1951 ; Bombay, 1956 ; Raj, 1960*]

Sol. The two sets of the opposite edges are

$$y+z=0, \quad x+z=0 \qquad ...(1)$$

and

$$x+y=0, \quad x+y+z=a \qquad ...(2)$$

Equation of the plane through the line (2) is

$$x+y+\lambda(x+y+z-a)=0 \qquad ...(3)$$

Writing (1) in the symmetrical form, we have

$$\frac{x}{1}=\frac{y}{1}=\frac{z}{-1} \qquad ...(4)$$

If (3) is parallel to (4), then

$$(1+\lambda)+(1+\lambda)-\lambda=0,$$

or

$$\lambda=-2.$$

$\therefore$ (3) becomes $(x+y)-2(x+y+z-a)=0$.

or

$$-x-y-2z+2a=0 \qquad ...(5)$$

Any point on (4) is $(r, r, -r)$.

$\therefore$ shortest distance between (1) and (2)

$=$length of the perpendicular from $(r, r, -r)$ on (5)

$$=\frac{-r-r+2r+2a}{\sqrt{1+1+4}}=\frac{2a}{\sqrt{6}}.$$

The six sets of edges are

$$\left.\begin{matrix} y+z=0 \\ z+x=0 \end{matrix}\right\} \quad ...(6), \qquad \left.\begin{matrix} x+y=0 \\ x+y+z=a \end{matrix}\right\} \quad ...(7)$$

$$\left.\begin{matrix} y+z=0 \\ x+y=0 \end{matrix}\right\} \quad ...(8), \qquad \left.\begin{matrix} x+y+z=a \\ z+x=0 \end{matrix}\right\} \quad ...(9)$$

$$\left.\begin{matrix} x+y+z=a \\ y+z=0 \end{matrix}\right\} \quad ...(10), \qquad \left.\begin{matrix} x+y=0 \\ z+x=0 \end{matrix}\right\} \quad ...(11)$$

Writing (6), (7), (8), (9), (10) and (11) in the symmetrical form, we have

$$\frac{x}{1}=\frac{y}{1}=\frac{z}{-1} \qquad ...(12)$$

$$\frac{x}{1}=\frac{y}{-1}=\frac{z-a}{0} \qquad ...(13)$$

$$\frac{x}{-1}=\frac{y}{1}=\frac{z}{-1} \qquad ...(14)$$

$$\frac{x}{1}=\frac{y-a}{0}=\frac{z}{-1} \qquad ...(15)$$

$$\frac{x}{1}=\frac{y}{-1}=\frac{z}{-1} \qquad ...(16)$$

$$\frac{x-a}{0}=\frac{y}{1}=\frac{z}{-1} \qquad ...(17)$$

Now, we shall find the equation of the line of shortest distance between (12) and (13).

Let the line of shortest distance meet (12) and (13) in points P_1 and P_2 whose coordinates are $(r, r, -r)$ and $(r', -r', a)$ respectively.

The direction ratios of $P_1 P_2$ are

$$r-r',\ r+r',\ -r-a.$$

$\because$ $P_1 P_2$ is perpendicular to both (12) and (13),

$$\therefore \quad (r-r')+(r+r')-(-r-a)=0$$

and

$$(r-r')-(r+r')=0.$$

Solving these equations, we have

$$r'=0,\ r=-\frac{a}{3}.$$

$\therefore$ P_1 and P_2 are $\left(-\dfrac{a}{3},\ \dfrac{-a}{3},\ \dfrac{-a}{3}\right)$ and (0, 0, 0).

$\therefore$ equation of the line of shortest distance between (12) and (13) is

$$\frac{x}{-a/3}=\frac{y}{-a/3}=\frac{z-a}{a/3-a},$$

or,

$$\frac{x}{1}=\frac{y}{1}=\frac{z-a}{2} \qquad ...(18)$$

Similarly, the shortest distances between (14), (15) and (16), (17) are respectively

$$\frac{x}{1}=\frac{y-a}{2}=\frac{z}{1} \qquad ...(19)$$

and

$$\frac{x-a}{2}=\frac{y}{1}=\frac{z}{1} \qquad ...(20)$$

The lines (18), (19) and (20) meet at the point

$$x=y=z=-a.$$

Ex. 2. Find the shortest distance between the lines

(*i*) $3x-9y+5z=0=x+y-z$

and $6x+8y+3z-13=0=x+2y+z-3.$ *(Poona, 1950)*

[**Ans.** $11/\sqrt{342}$.]

(*ii*) $2x+y-z=0=x-y+2z$

and $x+2y-3z-4=0=2x-3y+4z-5.$ *(Agra Engg., 1962)*

[**Ans.** $4/\sqrt{14}$.]

Ex. 3. Show that the equation to the plane containing the line

$$\frac{y}{b}+\frac{z}{c}=1,\ x=0\ ;$$

and parallel to the line

$$\frac{x}{a}-\frac{z}{c}=1,\ y=0$$

is

$$\frac{x}{a}-\frac{y}{b}-\frac{z}{c}+1=0. \qquad \textit{(Raj., 1963)}$$

If 2d be the shortest distance, prove that

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{d^2}.$$

(Allahabad, 1950 ; Punjab, 1955 S ; Agra, 1954 ; Raj., 1951 ; Poona, 1960, Banaras, 1960 ; Karnatak, 1955 ; Kashmir, 1951; Aligarh 1961)

Sol. Equation of any plane through the line

$$\frac{y}{b}+\frac{z}{c}=1,\ x=0$$

is
$$\frac{y}{b}+\frac{z}{c}-1+\lambda x=0 \quad \ldots(1)$$

If (1) is parallel to

$$\frac{x}{a}-\frac{z}{c}=1,\ y=0,$$

or,
$$\frac{x}{a}=\frac{z+c}{c}=\frac{y}{0}, \quad \ldots(2)$$

then
$$\lambda\,.\,a+0\,.\,\frac{1}{b}+c\,.\,\frac{1}{c}=0,$$

or
$$\lambda=-\frac{1}{a}.$$

$\therefore$ (1) becomes

$$\frac{y}{b}+\frac{z}{c}-1-\frac{1}{a}x=0,$$

or,
$$\frac{x}{a}-\frac{y}{b}-\frac{z}{c}+1=0. \quad \ldots(3)$$

The raquired shortest distance=the length of the perpendicular from any point $(a, 0, 0)$ of (2) on (3),

or,
$$2d=\frac{2}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}.$$

or,
$$\frac{1}{d^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}.$$

Type IX. Ex. 1. A square ABCD of diagonal 2a is folded along the diagonal AC so that the planes DAC and BAC are at right angles. Find the shortest distance between DC and AB.

(*Karnatak, 1956* ; *Punjab, 1951* ; *Poona, 1957*)

Sol. Let ABCD be a square of diagonal $2a$. Let it be folded along the diagonal AC so that the planes DAC and BAC are at right angles. Let O be the centre of the square. Let OAX be taken as the x-axis, O, the origin. Let OB and OD be taken as the axes of y and z respectively.

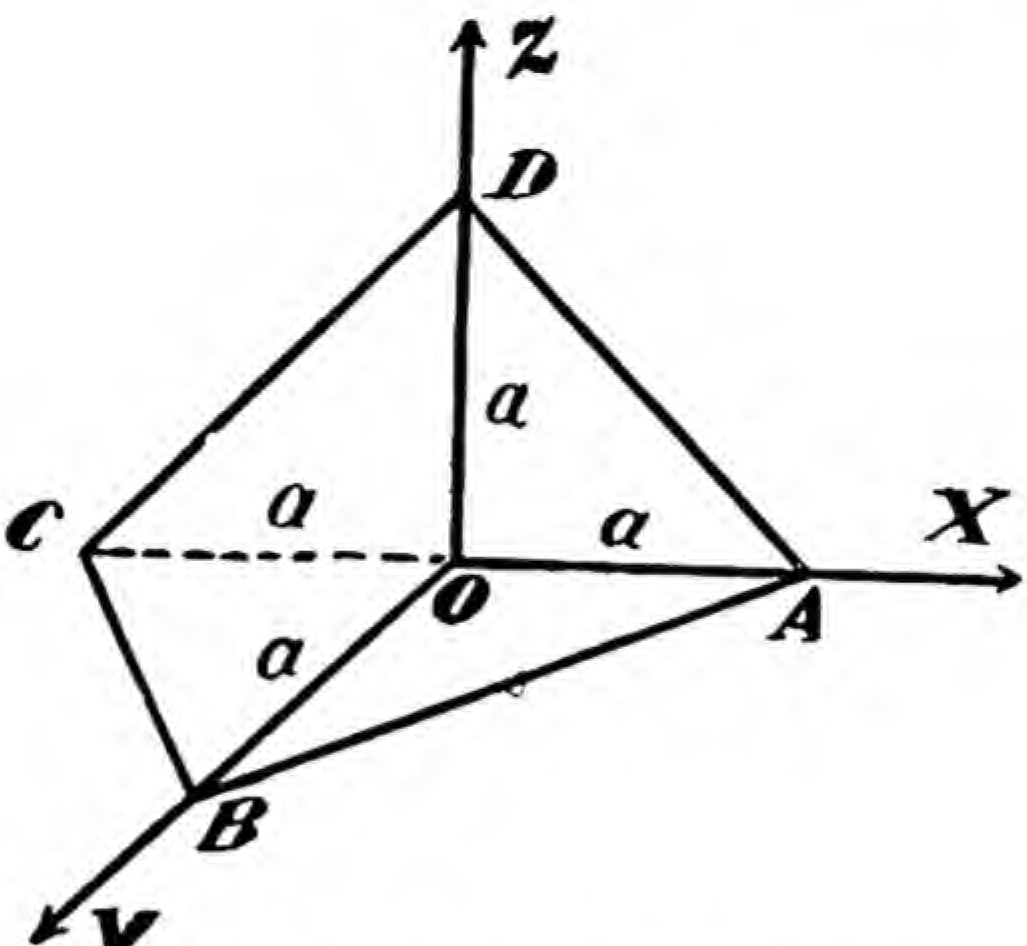
$\therefore$ coordinates of A, B, C, D are respectively $(a, 0, 0)$, $(0, a, 0)$, $(-a, 0, 0)$ and $(0, 0, a)$.

Equations of AB are

$$\frac{x-a}{a}=\frac{y}{-a}=\frac{z}{0} \quad \ldots(1)$$

and that of DC are

$$\frac{x}{a}=\frac{y}{0}=\frac{z-a}{a} \quad \ldots(2)$$



Equation of a plane through (2) and parallel (1) is

$$\begin{vmatrix} x & y & z-a \\ a & 0 & a \\ a & -a & 0 \end{vmatrix} = 0,$$

or, $$x(a^2)-y(-a^2)+(z-a)(-a^2)=0,$$

or, $$x+y-z+a=0, \qquad \ldots(3)$$

The required shortest distance = the length of the perpendicular from $(a, 0, 0)$ on (3)

$$=\frac{a+a}{\sqrt{1+1+1}}=\frac{2a}{\sqrt{3}}.$$

Ex. 2. Find the shortest distance between the diagonal of a rectangular solid and the edge it does not meet. (*Agra 1960 ; Delhi Engg., 1962*)

[**Ans.** $\dfrac{bc}{\sqrt{b^2+c^2}}, \dfrac{ca}{\sqrt{c^2+a^2}}, \dfrac{ab}{\sqrt{a^2+b^2}}$,

where a, b, c are the edges of the solid.]

SECTION VI

LENGTH OF THE PERPENDICULAR FROM A POINT ON A LINE

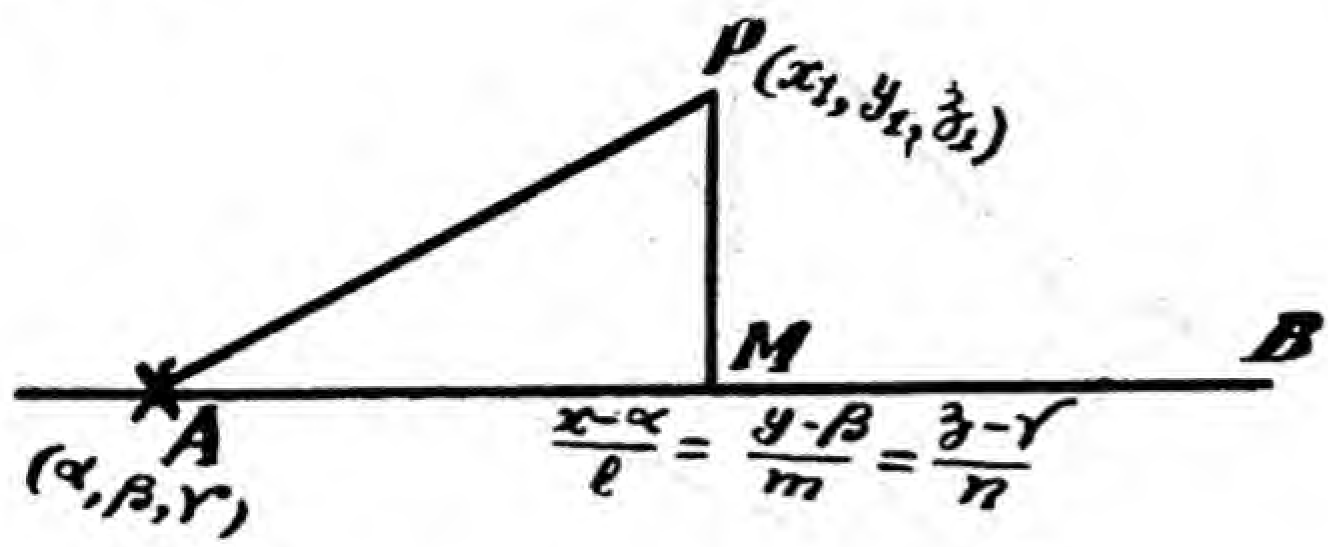
4·16. To find the length of the perpendicular from (x_1, y_1, z_1) on the line $\dfrac{x-\alpha}{l}=\dfrac{y-\beta}{m}=\dfrac{z-\gamma}{n}$, l, m, n being the direction cosines.

Let AB be the given line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad \ldots(1)$$

passing through the point A (α, β, γ). Let P be the point (x_1, y_1, z_1). Draw PM perpendicular to AB.

Join PA.



Now, AM = projection of AP on a line AB

$$=(x_1-\alpha)\,l+(y_1-\beta)m+(z_1-\gamma).n$$

Also, $AP=\sqrt{(x_1-\alpha)^2+(y_1-\beta)^2+(z_1-\gamma)^2}$.

From right-angled triangle AMP, we have

$$PM^2=AP^2-AM^2$$
$$=(x_1-\alpha)^2+(y_1-\beta)^2+(z_1-\gamma)^2-[(x_1-\alpha)l+(y_1-\beta)m+(z_1-\gamma)n]^2$$
$$=[(x_1-\alpha)^2+(y_1-\beta)^2+(z_1-\gamma)^2]\,[\mathbf{l}^2+\mathbf{m}^2+\mathbf{n}^2]$$
$$-[(x_1-\alpha)l+(y_1-\beta)m+(z_1-\gamma)n]^2 \text{ (Note this step.)}$$
$$=[(y_1-\beta)l-(x_1-\alpha)m]^2+[(z_1-\gamma)m-(y_1-\beta)n]^2$$
$$=[(x_1-\alpha)n-(z_1-\gamma)\,l]^2, \text{ using Lagrange's identity.}$$

$$\therefore \quad \mathbf{PM}=\sqrt{[(\mathbf{y}_1-\beta)\mathbf{l}-(\mathbf{x}_1-\alpha)\mathbf{m}]^2+[(\mathbf{z}_1-\gamma)\mathbf{m}-(\mathbf{y}_1-\beta)\mathbf{n}]^2+[\mathbf{x}_1-\alpha)\mathbf{n}-(\mathbf{z}_1-\gamma)\mathbf{l}]^2},$$

which is the required length of the perpendicular.

Caution. Here l, m, n are actual direction cosines.

Aid to memory. The successive terms in the result are the squares of the determinants $\begin{vmatrix} x_1-\alpha & y_1-\beta \\ \mathbf{l} & \mathbf{m} \end{vmatrix}$,

$\begin{vmatrix} \mathbf{y}_1-\beta & \mathbf{z}_1-\gamma \\ \mathbf{m} & \mathbf{n} \end{vmatrix}$ **and** $\begin{vmatrix} \mathbf{z}_1-\gamma & x_1-\alpha \\ \mathbf{n} & \mathbf{l} \end{vmatrix}$.

The required perpendicular distance is the square root of the sum of the squares of these determinants.

EXAMPLES IV (F)

Ex. 1. Find the pependicular distance from the point $(-1, 3, 9)$ of the line $$\frac{x-13}{5}=\frac{y+8}{-8}=\frac{z-3}{1}.$$ *(Delhi Hons., 1954).*

Sol. Let A be the point $(13, -8, 3)$ and P be the point $(-1, 3, 9)$. Draw PM perpendicular to the given line. Join AP.

$$\therefore \quad PM^2=AP^2-AM^2=(13+1)^2+(-8-3)^2+(3-9)^2$$
$$-\left[14\cdot\frac{5}{\sqrt{90}}+11\cdot\frac{8}{\sqrt{90}}-\frac{6}{\sqrt{90}}\right]^2$$
$$=196+121+36-\frac{23104}{90}=\frac{8666}{20}=\frac{4333}{45}$$
$$\therefore \quad PM=\sqrt{\frac{4333}{45}}.$$

Aliter

Any point on the line is $M\,(13+5R, -8-8R, 3+R)$

Let M be the foot of the perpendicular.

$\therefore$ direction ratios of PM are $14+5R, -11-8R, -6+R$

$\because$ it is perpendicular to the given line,

$\therefore$ $5(14+5R)-8(-11-8R)+(R-6)=0$,

or, $90R=-152$,

$\therefore$ $R=\dfrac{-152}{90}$

$\therefore$ Coordinates of M are

$$\left[\frac{41}{9}\cdot\frac{248}{45},\frac{59}{45}\right]$$

$$\therefore \quad PM=\sqrt{\left[\frac{41}{9}+1\right)^2+\left(\frac{248}{45}-3\right)^2+\left(\frac{59}{45}-9\right)^2}$$
$$=\sqrt{\frac{2500}{81}+\frac{12769}{81.25}+\frac{119716}{81.25}}=\sqrt{\frac{194985}{81.25}}=\sqrt{\frac{4333}{45}}.$$

Ex. 2. How far is the point $(4, 1, 1)$ from the line of intersection of the planes $x+y+z=4$ and $x-2y-z=4$? *(Punjab, 1961)*

[**Ans.** $3\sqrt{3}/\sqrt{14}$.]

Ex. 3. Find the distance of $A(1, -2, 3)$ from the line, PQ, through P, $(2, -3, 5)$, which makes equal angles with the axes.

(Patna, 1962 ; Bihar, 1960 S ; Pakistan (Punjab), 1953 S)

$\left[\textbf{Ans.}\ \sqrt{\frac{14}{3}}.\right]$

Ex. 4. A line through the origin makes angles α, β, γ with its projections on the coordinate planes, which are rectangular. The distances of any point (x, y, z) from the line and its projections are d, a, b, c. Prove that

$$d^2=(a^2-x^2)\cos^2\alpha+(b^2-y^2)\cos^2\beta+(c^2-z^2)\cos^2\gamma. \qquad \textit{(Agra, 1958)}$$

SECTION VII

EQUATIONS OF TWO NON-INTERSECTING LINES

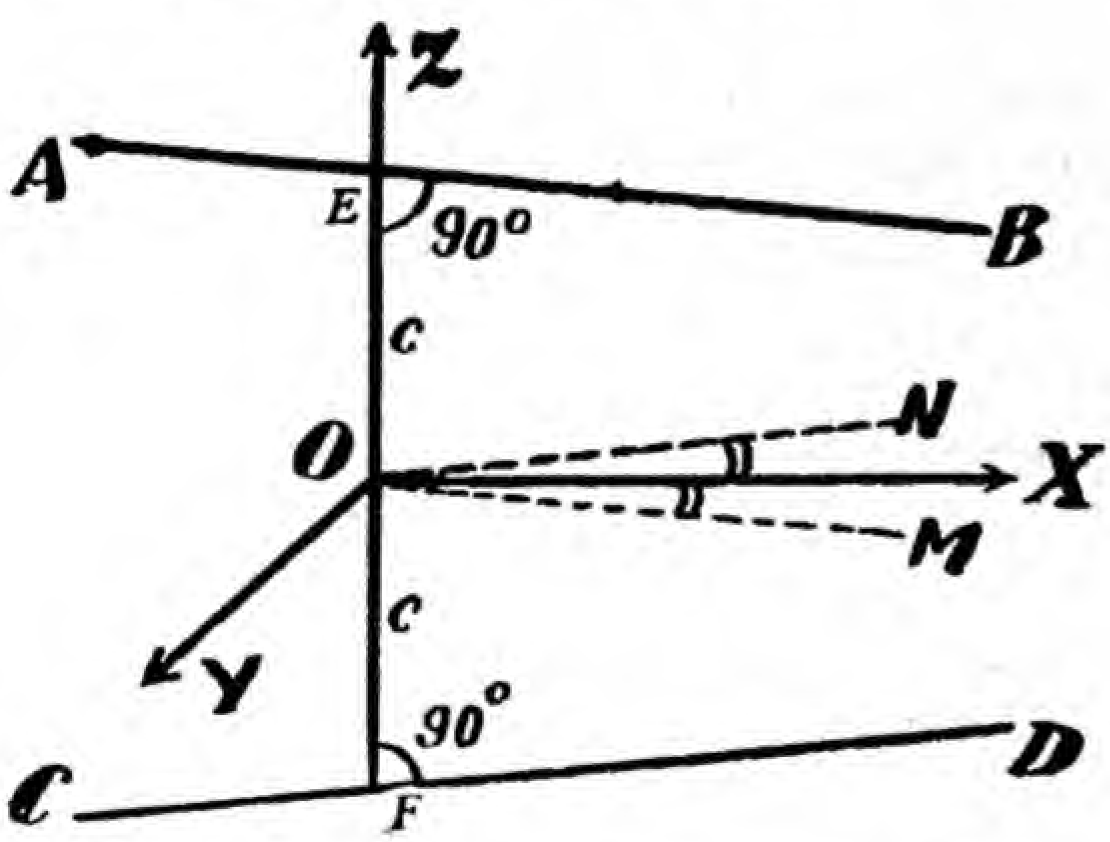
4·17. To show that, by a proper choice of axes, the equations of any two skew lines can be put into the form $y=mx$, $z=c$, and $y=-mx$, $z=-c$.

Proof. Let AB and CD be the two non-intersecting straight lines. Let $EF=2c$ be the shortest distance between them.

Through O, the middle point of EF, draw lines OM and ON parallel to AB and CD respectively.

Let us *choose O as the origin, the bisectors OX, OY of the angles between the lines OM and ON as the axes of x and y and FE as the z-axis.*

Let 2α be the angle between the given lines.



The line OM makes angles α, $\frac{\pi}{2}-\alpha$ and $\frac{\pi}{2}$ with the axes.

$\therefore$ its direction cosines are $\cos\alpha$, $\sin\alpha$, 0.

Also, the coordinates of E are $(0, 0, c)$.

Now, AB is a line passing through E and parallel to OM.

$\therefore$ its equations are $\dfrac{x-0}{\cos\alpha}=\dfrac{y-0}{\sin\alpha}=\dfrac{z-c}{0}$,

or, $$y=x\tan\alpha,\ z=c \qquad \ldots(1)$$

Again, the line ON makes angles $-\alpha$, $\frac{\pi}{2}+\alpha$, and $\frac{\pi}{2}$ with the axes.

$\therefore$ its direction cosines are $\cos\alpha$, $-\sin\alpha$, 0.

Also, the coordinates of F are $(0, 0, -c)$.

Now, CD is a line passing through F and parallel to ON.

$\therefore$ its equations are $\dfrac{x-0}{\cos\alpha}=\dfrac{y-0}{-\sin\alpha}=\dfrac{z+c}{0}$,

or, $$y=-x\tan\alpha,\ z=-c \quad \text{...(2)}$$

Putting $m=\tan\alpha$, the equations of AB and CD can be written as $\mathbf{y=mx, z=c}$; $\mathbf{y=-mx, z=-c}$. This proves the proposition.

Note. *Any point on the above lines can be taken as* (r, mr, c) and $(r', -mr', -c)$.

EXAMPLES IV (G)

Ex. 1. A line of constant length has its extremities on two fixed straight lines ; show that the locus of its middle point is an ellipse.

(Baroda, 1953 ; Delhi Hons., 1959, 1961 ; Jodhpur, 1963 ; Raj., 1956 ; Agra, 1952 ; Punjab T.D.C., 1965)

Sol Let the given lines be

$$\frac{x}{1}=\frac{y}{m}=\frac{z-c}{0} \quad \text{...(1)}$$

and

$$\frac{x}{1}=\frac{y}{-m}=\frac{z+c}{0} \quad \text{...(2)}$$

The coordinates of one extremity P of the transversal lying on (1) are (r, mr, c) and the coordinates of the other extremity P′ of the transversal lying on (2) are $(r', -mr', -c)$.

It is given that PP′ = constant $=2l$, say,

$$\therefore\ 4l^2=(r-r')^2+m^2(r'+r)^2+4c^2 \quad \text{...(3)}$$

Let $Q(\lambda, \mu, \nu)$ be the middle point of PP′.

$$\therefore\ 2\lambda=r+r',\quad 2\mu=m(r-r'),\quad \nu=0. \quad \text{...(4)}$$

From (3) and (4) we have, on eliminating r and r',

$$4l^2=\frac{4\mu^2}{m^2}+m^2.4\lambda^2+4c^2,\ \nu=0,$$

$\therefore$ locus of (λ, μ, ν) is

$$l^2=\frac{y^2}{m^2}+m^2x^2+c^2,\ z=0,$$

or $m^2x^2+\dfrac{y^2}{m^2}=l^2-c^2$, $z=0$, which is an ellipse in the xy-plane.

Ex. 2. AA′ is the shortest distance between two given lines, and B, B′ are variable points on them such that the volume AA′BB′ is constant. Prove that the locus of the middle point of BB′ is a hyperbola whose asymptotes are parallel to the lines. *(Raj., 1950 ; Calcutta Hons., 1963)*

Ex. 3. P, P′ are variable points on two given non-intersecting lines and PP′ is of constant length $2k$. Find the surface generated by PP′ *(Agra, 1959)*

$$\left[\textbf{Ans. } \frac{(cy-mxz)^2}{m^2}+(yz-cmx)^2=\frac{k^2-c^2}{c^2}(z^2-c^2)^2.\right]$$

Ex. 4. A point moves so that the line joining the feet of the perpendiculars from it to two given lines subtends a right angle at the middle point of their S.D. Show that its locus is a hyperbolic cylinder.

SECTION VIII

LINES INTERSECTING TWO GIVEN LINES OR THREE GIVEN LINES

4·18. To find the equations of any line intersecting two lines $\mathbf{u_1=o, v_1=o}$ and $\mathbf{u_2=o, v_2=o}$.

The given equations of the lines are

$$u_1=0=v_1 \qquad \ldots(1)$$

$$u_2=0=v_2 \qquad \ldots(2)$$

From Art. 3·22, the equation of any plane through the line (1) is

$$u_1+\lambda_1 v_1=0 \qquad \ldots(3)$$

and the equation of any plane through the second line is

$$u_2+\lambda_2 v_2=0 \qquad \ldots(4)$$

Let us now consider the line of intersection of the planes (3) and (4).

$\because$ this line of intersection of (3) and (4) lies in (3), it is coplanar with (1).

$\therefore$ it intersects (1), unless it is parallel to it.

Similarly, this line of intersection of (3) and (4) intersects (2).

Thus (3) and (4), *viz.* $\mathbf{u_1+\lambda_1 v_1=o, u_2+\lambda_2 v_2=o}$ taken together gives the required line.

Note. *The values of λ_1 and λ_2 are determined from the additional conditions of the problem.*

4·19. To find the locus of the line intersecting the given lines $\mathbf{u_1=o=v_1, u_2=o=v_2}$ and $\mathbf{u_3=o=v_3}$.

The given lines are $u_1=0=v_1$ ...(1)

$$u_2=0=v_2 \qquad \ldots(2)$$

$$u_3=0=v_3 \qquad \ldots(3)$$

The equations of line intersecting (1) and (2) are

$$u_1+\lambda_1 v_1=0,\ u_2+\lambda_2 v_2=0 \qquad \ldots(4) \text{ (Art. 4·18)}$$

If the line (4) intersects (3), we shall eliminate x, y, z between (3) and (4) and obtain a relation between λ_1 and λ_2 of the form

$$f(\lambda_1, \lambda_2)=0 \qquad \ldots(5)$$

The required locus is obtained by eliminating λ_1 and λ_2 between (4) and (5).

EXAMPLES IV (H)

Type I. Ex. 1. Find the direction cosines of the line through the origin which intersects each of the following two lines

$$\frac{x+1}{2}=\frac{y-3}{5}=-z-2.$$

and $\quad 3x-4y+z=x-2y-3z-1=0$. (*A.M.I.E., Nov. 1958, May 1961*)

Sol. The given lines are

$$5x-2y+11=0,\ y+5z+7=0. \qquad ...(1)$$

and

$$3x-4y+z=0,\ x-2y-3z-1=0. \qquad ...(2)$$

The equations of **any** line intersecting (1) and (2) are

$$\left.\begin{array}{r} 5x-2y+11+\lambda_1\,(y+5z+7)=0 \\ 3x-4y+z+\lambda_1(x-2y-3z-1)=0 \end{array}\right\} \qquad ...(3)$$

$\because$ (3) passes through (0, 0, 0),

$\therefore$ $$11+7\lambda_1=0$$

and $$\lambda_2=0,$$

or, $$\lambda_1=-\frac{11}{7},\ \lambda_2=0.$$

$\therefore$ (3) becomes $$35x-25y-55z=0,$$
$$3x-4y+z=0,$$

or $$7x-5y-11z=0,$$
$$3x-4y+z=0.$$

The direction ratios of this line are 49, 40, 13.

$\therefore$ direction cosines of this line are

$$\frac{49}{\sqrt{4170}},\ \frac{40}{\sqrt{4170}},\ \frac{13}{\sqrt{4170}}.$$

Ex. 2. Find the equations of the line through the origin which intersects each of the lines

$$\frac{x+1}{2}=\frac{y-2}{3}=\frac{z-6}{-1}$$

and $$\frac{x-2}{5}=\frac{y-3}{-3}=\frac{z-1}{2}. \qquad (A.M.I.E., 1961)$$

$$\left[\textbf{Ans}\quad \frac{x}{32}=\frac{y}{69}=\frac{z}{17}.\right]$$

Ex. 3. Find the equations to the line that intersects the lines $x+y+z=1$, $2x-y-z=2$; $x-y-z=3$, $2x+4y-z=4$, and passes through the point(1,1,1). (*Punjab (Pakistan), 1956 S ; Punjab 1965*)

$$\left[\textbf{Ans.}\ \frac{x-1}{0}=\frac{y-1}{1}=\frac{z-1}{3}.\right]$$

Ex. 4. Find the equations to the line drawn through the point $(1, 0, -1)$ and intersecting the lines $x=2y=2z$; $3x+4y=1$, $4x+5z=2$. (*Delhi Engg., 1963*)

[**Ans.** $x-3y+z=0$, $17x+12y+10z-7=0$.]

Type II. Ex. 1. Prove that the locus of a line which meets the lines

$$\mathbf{y=\pm mx,\ z=\pm c\ ;}$$

and the circle

$$\mathbf{x^2+y^2=a^2,\ z=0 \text{ is}}$$

$$\mathbf{c^2m^2(cy-mxz)^2+c^2(yz-cmx)^2=a^2m^2(z^2-c^2)^2.}$$

(*Delhi Hons., 1963 ; Kashmir, 1954 ; Raj. 1961 ; Punjab. 1962S ; Agra, 1963*)

Sol. Equations of any line intersecting the given lines

$$y-mx=0,\quad z-c=0 \qquad ...(1)$$

and

$$y+mx=0,\quad z+c=0 \qquad ...(2)$$

are

$$y-mx+\lambda_1(z-c)=0,\ y+mx+\lambda_2(z+c)=0 \qquad ...(3)$$

This meets $z=0$, where $y-mx-\lambda_1 c=0$,

$$y+mx+\lambda_2 c=0.$$

$\therefore$
$$y=\frac{c}{2}\cdot(\lambda_1-\lambda_2)$$

and
$$x=-\frac{c}{2m}(\lambda_1+\lambda_2).$$

The line (3) intersects the circle $x^2+y^2=a^2$, $z=0$,

if
$$\frac{c^2}{4m^2}(\lambda_1+\lambda_2)^2+\frac{c^2}{4}(\lambda_1-\lambda_2)^2=a^2,$$

or
$$\text{if } c^2(\lambda_1+\lambda_2)^2+c^2m^2(\lambda_1-\lambda_2)^2=4a^2m^2 \qquad ..(4)$$

The required locus of (3) is obtained by eliminating λ_1 and λ_2 between (3) and (4).

$\therefore$ the equation of the required locus is

$$c^2\left[\frac{y-mx}{z-c}+\frac{y+mx}{z+c}\right]^2+c^2m^2\left[\frac{y+mx}{z+c}-\frac{y-mx}{z-c}\right]^2=4a^2m^2,$$

or
$$\frac{c^2}{(z^2-c^2)^2}\left[yz-mxc\right]^2+\frac{c^2m^2}{(z^2-c^2)^2}(yc-mxz)^2=a^2m^2,$$

or,
$$c^2(yz-mxc]^2+c^2m^2(yc-mxz)^2=a^2m^2(z^2-c^2)^2.$$

Aliter.

Let the line intersect the given lines at

at
$$P(r,\ mr,\ c) \text{ and } P'(r',\ -mr',\ -c).$$

$\therefore$ equations of PP′ are

$$\frac{x-r}{r-r'}=\frac{y-mr}{m(r+r')}=\frac{z-c}{2c} \qquad ...(1)$$

It meets the circle $x^2+y^2=a^2$, $z=0$, where

$$x=r-\tfrac{1}{2}(r-r')=\tfrac{1}{2}(r+r'),\ y=mr-\tfrac{1}{2}m(r+r')$$
$$=\tfrac{1}{2}m(r-r').$$

$\therefore$
$$x^2+y^2=\tfrac{1}{4}[(r+r')^2+m^2(r-r')^2],$$

or
$$a^2=\tfrac{1}{4}[(r+r')^2+m^2(r-r')^2] \qquad ...(2)$$

Eliminating r and r' between (1) and (2), we get the required locus.

From (1), we have

$$\frac{z-c}{2c}=\frac{m(x-r)+y-mr}{m(r-r')+m(r+r')}=\frac{m(x-r)-(y-mr)}{m(r-r')-m(r+r')}.$$

or,
$$\frac{z-c}{2c}=\frac{mx+y-2mr}{2mr}=\frac{mx-y}{-2mr'}.$$

$\therefore$
$$r=\frac{c(mx+y)}{m(z+c)}$$

and $$r'=\frac{-c(mx-y)}{m(z-c)}.$$

$\therefore$ (2) gives

$$\frac{c^2}{m^2(z^2-c^2)^2}(yz-mxc)^2+\frac{c^2}{(z^2-c^2)^2}(mxz-yc)^2=a^2,$$

or, $$c^2m^2(cy-mxz)^2+c^2(yz-cmx)^2=a^2m^2(z^2-c^2)^2.$$

Ex. 2. Find the surface generated by a line which intersects the lines $y=z=a$; $x+3z=a$, $y+z=a$, and is parallel to the plane $x+y=0$.

(Allahabad, 1959)

[**Ans.** $(x+y)(y+z)=2a(x+z)$.]

Ex. 3. A variable line intersects OX and the curve $x=y$, $y^2=cz$, and is parallel to the plane YOZ. Prove that it generates the paraboloid $xy=cz$.

(Allahabad, 1960)

Ex. 4. Find the surface generated by a line which intersects two given lines and is parallel to a given plane. *(Delhi Hons., 1963 ; Punjab 1963)*

[**Ans.** $u(cy-mxz)+vm(cmx-yz)-wm(z^2-c^2)=0$, where the given lines are $y=mx$, $z=c$; $y=-mx$, $z=-c$ and the plane is $ux+vy+wz+d=0$.]

Type III. Ex. 1. Prove that the lacus of a variable line which intersects the three given lines $y=mx$, $z=c$; $y=-mx$, $z=-c$; $y=z$, $mx=-c$; is the surface $y^2-m^2x^2=z^2-c^2$.

(Agra, 1954 ; Jodhpur, 1964 ; Delhi Hons., 1961 ; Vikram 1964)

Sol. Equations of any line intersecting first two lines are

$$\left.\begin{aligned}(y-mx)+\lambda_1(z-c)=0\\(y+mx)+\lambda_2(z+c)=0\end{aligned}\right\}\qquad\ldots(1)$$

Putting $mx=-c$, or, $x=-\dfrac{c}{m}$ in (1), we have

$$y+c+\lambda_1 z-\lambda_1 c=0,$$
$$y-c+\lambda_2 z+\lambda_2 c=0$$

$$\therefore\quad y=\frac{c(\lambda_1+\lambda_2)-2\lambda_1\lambda_2 c}{\lambda_1-\lambda_2},\quad z=\frac{-2c+\lambda_1 c+\lambda^2 c}{\lambda_1-\lambda_2}.$$

$\because$ (1) intersects the line $y=z$, $mx=-c$,

$$\therefore\quad \frac{c(\lambda_1+\lambda_2)-2\lambda_1\lambda_2 c}{\lambda_1-\lambda_2}=\frac{-2c+\lambda_1 c+\lambda_2 c}{\lambda_1-\lambda_2},$$

or, $$c(\lambda_1+\lambda_2)-2\lambda_1\lambda_2 c=-2c+\lambda_1 c+\lambda_2 c,$$

or, $$\lambda_1\lambda_2=1\qquad\ldots(2)$$

The locus required is obtained by elliminating λ_1 and λ_2 between (1) and (2).

$$\therefore\quad \frac{y-mx}{z-c}\cdot\frac{y+mx}{z+c}=0,$$

or, $$y^2-m^2x^2=z^2-c^2.$$

Ex. 2. Show that the equation of any line which meets the three lines $x=a$, $y=0$; $y=a$, $z=0$; $z=a$, $x=0$ can be written as

$$x-a=\lambda y,\ z(\lambda+1)+\lambda(y-a)=0.$$

Show further that this line generates the surface

$$yz+zx+xy-ax-ay-az+a^2=0.\qquad \textit{(Bombay, 1960)}$$

Ex. 3. Find the equation of the surface generated by a straight line which intersects the three lines $y-2z=0,\ x=a;\ z-2x=0,\ y=a,$
$x-2y=0,\ z=a.$ *(Allahabad, 1958)*

[**Ans.** $2x^2+2y^2+2z^2-5yz-5zx-5xy+5ax+5ay+5az-7a^2=0.$]

Ex. 4. Prove that the locus of the lines which intersect the three lines
$y-z=1,\ x=0;\ z-x=1,\ y=0;$
$x-y=1,\ z=0$ is $x^2+y^2+z^2-2yz-2zx-2xy=1.$
(Allahabad, 1962; Agra, 1955)

Ex. 5. Prove that all lines which intersect the lines
$y=mx,\quad z=c;\quad y=-mx,\quad z=-c;$
and the x-axis, lie on the surface $mxz=cy$.

SECTION IX

INTERSECTION OF THREE PLANES

4·20. A useful notation.

Let three planes be $a_1x+b_1y+c_1z+d_1=0,$
$a_2x+b_2y+c_2z+d_2=0,$
$a_3x+b_3y+c_3z+d_3=0.$

Let us write down the coefficients of x, y, z and the constant terms in (1), (2) and (3) and enclose them between two double vertical bars. We have

$$\left\|\begin{matrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{matrix}\right\|.$$

Let us denote it by $\triangle$.

Let $\triangle_1, \triangle_2, \triangle_3, \triangle_4$ be the determinants obtained by deleting the first, second, third and fourth columns in succession in $\triangle$.

4·21. Nature of intersection of three planes.

1. To find the conditions that the planes

$a_rx+b_ry+c_rz+d_r=0$, $r=1, 2, 3$ may intersect in a straight line.

Let the given planes be $a_1x+b_1y+c_1z+d_1=0$ ...(1)
$a_2x+b_2y+c_2z+d_2=0$ ...(2)
$a_3x+b_3y+c_3z+d_3=0$ ...(3)

Equations of any plane through the line of intersection of (1) and (2) is

$$a_1x+b_1y+c_1z+d_1+\lambda(a_2x+b_2y+c_2z+d_2)=0$$

or $$(a_1+\lambda a_2)x+(b_1+\lambda b_2)y+(c_1+\lambda c_2)z+(d_1+\lambda d_2)=0 \quad ...(4)$$

If (1), (2), (3), intersect in a line, then

(3) must be identical with (4) for some value of λ. Comparing coefficients of like terms in (3) and (4), we have

$$\frac{a_1+\lambda a_2}{a_3}=\frac{b_1+\lambda b_2}{b_3}=\frac{c_1+\lambda c_2}{c_3}=\frac{d_1+\lambda d_2}{d_3}=k, \text{ say.}$$

$$\therefore \quad a_1+\lambda a_2-ka_3=0 \qquad \ldots(5)$$

$$b_1+\lambda b_2-kb_3=0 \qquad \ldots(6)$$

$$c_1+\lambda c_2-kc_3=0 \qquad \ldots(7)$$

$$d_1+\lambda d_2-kd_3=0 \qquad \ldots(8)$$

Eliminating λ and k between these equations, taken three at a time, we have

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0, \quad \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}=0, \quad \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}=0.$$

and

$$\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}=0,$$

or, $\quad \triangle_4=0, \ \triangle_1=0, \ \triangle_2=0, \ \triangle_3=0,$

or, $\quad \triangle_1=\triangle_2=\triangle_3=\triangle_4=0$, which are the required conditions.

But out of these four conditions only **two** are **independent**, because if two planes have two points in common, they have a common line of intersection and for this fact only two conditions are required.

Note. *It can be proved algebraically that if two of these determinants vanish, the other two must also vanish, i.e., if* $\triangle_4=0=\triangle_1$, *then* $\triangle_3=\triangle_2=0$ *also.*

Aliter.

The line of intersection of (1) and (2) in the symmetrical form is

$$\frac{x-\dfrac{b_1d_2-b_2d_1}{a_1b_2-a_2b_1}}{b_1c_2-b_2c_1}=\frac{y-\dfrac{a_2d_1-a_1d_2}{a_1b_2-a_2b_1}}{a_2c_1-a_1c_2}=\frac{z}{a_1b_2-a_2b_1} \qquad \ldots(A)$$

If (1), (2) and (3) intersect in a line, then the line (A) lies completely in the plane (3), the conditions for which are that (*i*) (A) must be parallel to (3), *i.e.* , $\triangle_4=0$ and (*ii*) a point of (A), *viz.*,

$$\left(\frac{b_1d_2-b_2d_1}{a_1b_2-a_2b_1},\ \frac{a_2d_1-a_1d_2}{a_1b_2-a_2b_1},\ 0\right) \text{ must satisfy (3),}$$

i.e., $$a_3\left(\frac{b_1d_2-b_2d_1}{a_1b_2-a_2b_1}\right)+b_3\left(\frac{a_2d_1-a_1d_2}{a_1b_2-a_2b_1}\right)+d_3=0,$$

or, $$a_3(b_1d_2-b_2d_1)+b_3(a_2d_1-a_1d_2)+d_3(a_1b_2-a_2b_1)=0$$

or, $$a_1(b_2d_3-b_3d_2)-a_2(b_1d_3-b_3d_1)+a_3(b_1d_2-b_2d_1)=0,$$

or, $$\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}=0,$$

or $$\Delta_4=0.$$

$\therefore$ the required conditions are $\Delta_3=0=\Delta_4$.

II. To find the condition that the planes
$$\mathbf{a_rx+b_ry+c_rz+d_r=0,\ r=1,\ 2,\ 3}$$
may intersect in a point.

Solving the given equations, we have
$$\frac{x}{\Delta_1}=\frac{-y}{\Delta_2}=\frac{z}{\Delta_3}=\frac{-1}{\Delta_4}.$$

$\therefore$ $$x=-\Delta_1/\Delta_4,\ y=\Delta_2/\Delta_4,\ z=-\Delta_3/\Delta_4.$$

$\because$ the planes intersect in a point, the coordinates of their point of intersection must be finite, which requires that $\Delta_4\neq 0$.

This is the required condition.

Aliter. The given planes will meet in a point if the line of intersection of any two of them is not parallel to the third.

Let l, m, n be the direction ratios of the line of intersection of the first two planes.

$\therefore$ $$a_1l+b_1m+c_1n=0$$

and $$a_2l+b_2m+c_2n=0.$$

$\therefore$ $$\frac{l}{b_3c_2-b_2c_1}=\frac{m}{a_2c_1-a_1c_2}=\frac{n}{a_1b_2-a_2b_1}.$$

This line of intersection will not be parallel to the third plane if
$$a_3(b_1c_2-b_2c_1)+b_3(a_2c_1-a_1c_2)+c_3(a_1b_2-a_2b_1)\neq 0,$$

or $$a_1(b_2c_3-b_3c_2)-a_2(b_1c_3-b_3c_1)+a_3(b_1c_2-b_2c_1)\neq 0,$$

or, $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}\neq 0,$$

or $\Delta_4\neq 0$, which is the required condition.

Note. *The point of intersection of the three planes can be obtained by solving their equations simultaneously in x, y, z.*

III. To find the conditions that the three planes

$$\mathbf{a_r x + b_r y + c_r z + d_r = 0, \ \ r = 1, 2, 3}$$

may form a triangular prism.

The line of intersection of the first two planes is

$$\frac{x-\dfrac{b_1d_2-b_2d_1}{a_1b_2-a_2b_1}}{b_1c_2-b_2c_1}=\frac{y-\dfrac{a_2d_1+a_1d_2}{a_1b_2-a_2b_1}}{a_2c_1-a_1c_2}=\frac{z}{a_1b_2-a_2b_1}$$

The three planes will *form a triangular prism if this line of intersection is parallel to the third plane.*

This requires that this line is perpendicular to the normal to the third plane and its point

$$\left(\frac{b_1d_2-b_2d_1}{a_1b_2-a_2b_1},\ \frac{a_2d_1-a_1d_2}{a_1b_2-a_2b_1},\ 0\right)$$

does not lie on the third plane.

$$\therefore \quad a_3(b_1c_2-b_2c_1)+b_3(c_1a_2-c_2a_1)+c_3(a_1b_2-a_2b_1)=0$$

and
$$a_3\left(\frac{b_1d_2-b_2d_1}{a_1b_2-a_2b_1}\right)+b_3\left(\frac{a_2d_1-a_1d_2}{a_1b_2-a_2b_1}\right)+d_3\neq 0,$$

or,
$$a_3(b_1d_2-b_2d_1)+b_3(a_2d_1-a_1d_2)+d_3(a_1b_2-a_2b_1)\neq 0,$$

i.e.,
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0$$

and
$$\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}\neq 0,$$

or
$$\triangle_4=0$$

and
$$\triangle_3\neq 0,$$

which are the required conditions.

EXAMPLES IV (I)

Type I. Ex. 1. Prove that the planes

$$\mathbf{x+ay+(b+c)z+d=0,}$$
$$\mathbf{x+by+(c+a)z+d=0,}$$
$$\mathbf{x+cy+(a+b)z+d=0}$$

pass through one line.

Sol. Here $\Delta_4=\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$

$$=\begin{vmatrix} 1 & a & a+b+c \\ 1 & b & a+b+c \\ 1 & c & a+b+c \end{vmatrix}.$$

on adding the second column to the third.

$$=(a+b+c)\begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix}=0.$$

Also,

$$\Delta_3=\begin{vmatrix} 1 & a & d \\ 1 & b & d \\ 1 & c & d \end{vmatrix}=d\begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix}=0,$$

$\therefore$ $\Delta_4=0=\Delta_3$.

$\therefore$ the given planes intersect in a straight line.

Ex. 2. Show that the planes

$$cy-bz=l,$$
$$az-cx=m,$$
$$bx-ay=n$$

intersect in a line if $al+bm+cn=0$.

Ex. 3. Prove that the planes

(*i*)
$$2x+5y+3z=0,$$
$$x-y+4z=2,$$
$$7y-5z+4=0$$

pass through one line. *(Bihar, 1961 S)*

(*ii*)
$$2x-3y-7z=0,$$
$$3x-14y-13z=0,$$
$$8x-31y-33z=0$$

pass through one line. *(North Bengal, 1964)*

Type II. Ex. 1. Prove that the planes

$$\mathbf{x=cy+bz},$$
$$\mathbf{y=az+cx},$$
$$\mathbf{z=bx+ay}$$

pass through one line if

$$\mathbf{a^2+b^2+c^2+2abc=1},$$ *(Delhi Engg., 1963)*

and show that the line of intersection then is

$$\frac{\mathbf{x}}{\sqrt{\mathbf{1-a^2}}}=\frac{\mathbf{y}}{\sqrt{\mathbf{1-b^2}}}=\frac{\mathbf{z}}{\sqrt{\mathbf{1-c^2}}}.$$

(I.A.S., 1953; Nagpur T.D.C., 1962; Bombay, 1953)

Sol. The given planes are

$$x-cy-bz=0 \quad \ldots(1)$$

$$cx-y+az=0 \quad \ldots(2)$$

and

$$bx+ay-z=0 \quad \ldots(3)$$

Let l, m, n be the direction ratios of the line of intersection of (1) and (2).

$\therefore$
$$l-cm-bn=0,$$
$$cl-m+an=0.$$

$\therefore$
$$\frac{l}{ac+b}=\frac{m}{bc+a}=\frac{n}{1-c^2} \quad \ldots(4)$$

$\because$ (1) and (2) pass through origin,

$\therefore$ their line of intersection will also pass through the origin.

$\therefore$ equations of the line of intersection of (1) and (2) are

$$\frac{x}{ac+b}=\frac{y}{bc+a}=\frac{z}{1-c^2} \quad \ldots(5)$$

If (1), (2) and (3) intersect in a line, then (5) lies in (3).

The point (0, 0, 0) of (5) satisfies (3).

$\therefore$ the condition is $b(ac+b)+a(bc+a)-(1-c^2)=0$,

or,
$$a^2+b^2+c^2+2abc=1 \quad \ldots(6)$$

The direction ratios of the line (5) can be written as

$\sqrt{a^2c^2+b^2+2abc}$, $\sqrt{b^2c^2+a^2+2abc}$, $1-c^2$,

or, $\sqrt{a^2c^2+1-c^2-a^2}$, $\sqrt{b^2+c^2+1-b^2-c^2}$, $1-c^2$, using (6),

or, $\sqrt{(1-c^2)(1-a^2)}$, $\sqrt{(1-b^2)(1-c^2)}$, $1-c^2$,

or, $\sqrt{1-a^2}$, $\sqrt{1-b^2}$, $\sqrt{1-c^2}$.

$\therefore$ the line of intersection is

$$\frac{x}{\sqrt{1-a^2}}=\frac{y}{\sqrt{1-b^2}}=\frac{z}{\sqrt{1-c^2}}.$$

Ex. 2. Show that the planes $x=y \sin\psi+z\sin\phi$, $y=z\sin\theta+x\sin\psi$, $z=x\sin\phi+y\sin\theta$ intersect in the line

$$\frac{x}{\cos\theta}=\frac{y}{\cos\phi}=\frac{z}{\cos\psi}, \text{ if } \theta+\phi+\psi=\frac{\pi}{2}.$$

Ex. 3. The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meet the axes in A, B, C. Prove that the planes through the axes and the internal bisectors of the angles of the triangle ABC pass through the line

$$\frac{x}{a\sqrt{b^2+c^2}}=\frac{y}{b\sqrt{c^2+a^2}}=\frac{z}{c\sqrt{a^2+b^2}}.$$

Ex. 4. The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the axes OX, OY, OZ which are rectangular in A, B, C. Find the equations of BC. Prove that the planes

through the axes and perpendicular to BC, CA, AB pass through the line $ax=by=cz$. **Find the coordinates of the orthocentre of the** $\triangle$**ABC.**

(Punjab, 1959S)

$$\left[\text{Ans. } \frac{x}{0}=\frac{y}{b}=\frac{z-c}{-c}\ ; \left(\frac{a^{-1}}{a^{-2}+b^{-2}+c^{-2}}, \frac{b^{-1}}{a^{-2}+b^{-2}+c^{-2}}, \frac{c^{-1}}{a^{-2}+b^{-2}+c^{-2}}\right)\right]$$

Ex. 5. Prove that the planes $12x-15y+16z-28=0$, $6x+6y-7z-8=0$, $2x+35y-39z+12=0$ have a common line of intersection; prove also that the point in which the line $\frac{x-1}{3}=\frac{-y}{2}=z-3$ meets the third plane is equidistant from the other two planes. *(A.M.I.E., Nov., 1958 ; Raj. Engi., 1963)*

Type III. Ex. 1. Prove that the three planes $2x+y+z=3$, $x-y+2z=4$, $x+z=2$, form a triangular prism, and find the area of a normal section of the prism. *(A.M.I.E., 1958 ; Raj. Eng., 1957 ; Raj., 1955)*

Sol. The given planes are

$$2x+y+z=3 \quad ...(1), \qquad x-y+2z=4 \quad ...(2)$$

and

$$x+z=2 \quad ...(3).$$

Here $\Delta_4=\begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 0 & 1 \end{vmatrix}=2(-1)-(1)+3=0,$

Also, $\Delta_3=\begin{vmatrix} 2 & 1 & -3 \\ 1 & -1 & -4 \\ 1 & 0 & -2 \end{vmatrix}=2(2)-1(-2)+1(-4-3)$

$=4+2-7=-1$, which is not zero.

$\therefore$ the given planes form a triangular prism.

Let l, m, n be the direction ratios of the line of intersection of (1) and (2).

$\therefore \quad 2l+m+n=0$ and $l-m+2n=0$

$\therefore \quad \frac{l}{1}=\frac{m}{-1}=\frac{n}{-1}.$

To find a point P on this line, put $z=0$ in (1) and (2).

$\therefore \quad 2x+y=3$ and $x-y=4$,

or, $\quad x=\frac{7}{3}, y=-\frac{5}{3}$.

$\therefore$ Coordinates of P are $(\frac{7}{3}, -\frac{5}{3}, 0)$.

Let PQR be the normal section of the prism by the plane through P perpendicular to the line of intersection of (1) and (2).

Any plane perpendicular to this line is

$$x-y-z=\lambda.$$

$\because$ it passes through P, $\therefore$ $\frac{7}{3}-(-\frac{5}{3})-0=\lambda$, or, $\lambda=4$

$\therefore$ equation of this plane is $x-y-z=4$ ...(4)

Q is the point of intersection of (1), (3) and (4),

$\therefore$ coordinates of Q are $(\frac{7}{3}, -\frac{4}{3}, -\frac{1}{3})$.

R is the point of intersection (2), (3) and (4),

$\therefore$ coordinates of R are $(2, -2, 0)$.

$$\text{Now, QR}=\sqrt{\left(\frac{7}{3}-2\right)^2+\left(-\frac{4}{3}+2\right)^2+\frac{1}{9}}$$

$$\sqrt{\frac{1}{9}+\frac{4}{9}+\frac{1}{9}}=\sqrt{\frac{2}{3}}.$$

Also, PM=perpendicular from P on plane (3)

$$=\frac{\frac{7}{3}+0-2}{\sqrt{1+1}}=\frac{1}{3\sqrt{2}}.$$

$\therefore$ area of the triangle PQR$=\frac{1}{2}$QR$\times$PM

$$=\frac{1}{2}\cdot\sqrt{\frac{2}{3}}\cdot\frac{1}{3\sqrt{2}}=\frac{1}{6\sqrt{3}}.$$

Aliter.

Direction ratios of PQ are $0, -1, 1$.

Direction ratios of PR are $1, 1, 0$.

Now $\cos QPR=-\frac{1}{2}$

$\therefore$ $\sin QPR=\frac{\sqrt{3}}{2}$.

Also, $$PQ=\sqrt{\frac{1}{9}+\frac{1}{9}}=\sqrt{\frac{2}{9}}$$

and $$PR=\sqrt{\frac{1}{9}+\frac{1}{9}}=\sqrt{\frac{2}{9}}.$$

$\therefore$ area of $\triangle PQR=\frac{1}{2}PQ.PR\sin QPR=\frac{1}{2}.\frac{\sqrt{2}}{3}.\frac{\sqrt{2}}{3}.\frac{\sqrt{3}}{2}=\frac{1}{6\sqrt{3}}$.

The coordinates of P, Q, R are $\left(\frac{7}{3}, \frac{-5}{3}, 0\right), \left(\frac{7}{3}, -\frac{4}{3}, -\frac{1}{3}\right)$, $(2, -2, 0)$.

Let A_x, A_y, A_z be the projection of $\triangle PQR$ on the coordinate planes of yz, zx and xy.

The coordinates of the points of projection in the yz plane of the three vertices are

$$\left(0, -\frac{5}{3}, 0\right), \left(0, -\frac{4}{3}, -\frac{1}{3}\right), \text{ and } (0, -2, 0).$$

In two dimensions, these points are

$$\left(-\frac{5}{3}, 0\right), \left(-\frac{4}{3}, -\frac{1}{3}\right) \text{ and } (-2, 0).$$

$$\therefore \quad A_x = \frac{1}{2} \begin{vmatrix} -\frac{5}{3} & 0 & 1 \\ -\frac{4}{3} & -\frac{1}{3} & 1 \\ -2 & 0 & 1 \end{vmatrix}$$

$$= \frac{1}{2} \cdot \left[\frac{5}{9} - \frac{2}{3}\right] = \frac{1}{2} \cdot \left(-\frac{1}{9}\right) = -\frac{1}{18}.$$

Similarly, $$A_y = \frac{1}{2} \begin{vmatrix} \frac{7}{3} & 0 & 1 \\ \frac{7}{3} & -\frac{1}{3} & 1 \\ 2 & 0 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[-\frac{7}{9} + \frac{2}{3}\right] = \frac{1}{2} \cdot \left(-\frac{1}{9}\right) = -\frac{1}{18}.$$

$$A_z = \frac{1}{2} \begin{vmatrix} \frac{7}{3} & -\frac{5}{3} & 1 \\ \frac{7}{3} & -\frac{4}{3} & 1 \\ 2 & -2 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[\frac{7}{3}\left(-\frac{4}{3}+2\right) + \frac{5}{3}\left(\frac{7}{3}-2\right) + \left(-\frac{14}{3}+\frac{8}{3}\right)\right]$$

$$= \frac{1}{2}\left[\frac{14}{9} + \frac{5}{9} - 2\right] = \frac{1}{2} \cdot \frac{1}{9} = \frac{1}{18}.$$

Now $$A = \sqrt{A^2_x + A^2_y + A^2_z} = \sqrt{\left(\frac{1}{18}\right)^2 + \left(\frac{1}{18}\right)^2 + \left(\frac{1}{18}\right)^2}$$

$$= \frac{\sqrt{3}}{18} = \frac{1}{6\sqrt{3}}.$$

Ex. 2. Show that the planes

$$2x+3y+4z=6,$$
$$3x+4y+5z=20,$$
$$x+2y+3z=2$$

form a triangular prism. Find the equations of the edges and the equation of the right section passing through $(4, 5, -1)$. Find also the area of the right section.

(Bombay, 1959)

$$\left[\text{Ans. } \frac{x-16}{1} = \frac{y+7}{-2} = \frac{z}{1}\ ;\ \frac{x-6}{1} = \frac{y+2}{-2} = \frac{z}{1}\ ;\right.$$
$$\left.\frac{x-36}{1} = \frac{y+22}{-2} = \frac{z}{1}\ ;\ x-2y+z+7=0\ ;\ 25/\sqrt{6}.\right]$$

Ex. 3. Show that the planes

$$x-y-z+2=0,$$
$$3x-6y-5z+3=0$$
$$6x-9y-8z+3=0$$

form a triangular prism. Find the area and the lengths of the edges of its normal section.

$$\left[\textbf{Ans.}\quad 2\sqrt{5}\,;\quad \frac{3}{7}\sqrt{14},\ \frac{3}{7}\sqrt{42},\ \frac{1}{7}\sqrt{2534}.\right]$$

MISCELLANEOUS (REVISION) EXAMPLES ON CHAPTER IV

1. Find the equation to the plane through the points $(2, -1, 0)$ and $(3, -4, 5)$ parallel to the line

$$2x=3y=4z$$

(Karnatak, 1958)

[**Ans.** $29x-27y-22z-85=0.$]

2 Show that the shortest distanee between the lines

$$\frac{x-x_1}{\cos\alpha_1}=\frac{y-y_1}{\cos\beta_1}=\frac{z-z_1}{\cos\gamma_1}$$

and

$$\frac{x-x_2}{\cos\alpha_2}=\frac{y-y_2}{\cos\beta_2}=\frac{z-z_2}{\cos\gamma_2}$$

meets the first line at a point whose distance from

(x_1, y_1, z_1) is $[\Sigma(x_1-x_2)(\cos\alpha_1-\cos\theta\cos\alpha_2)]\operatorname{cosec}^2\theta$,

where θ is the angle between the lines. *(Karnatak, 1961)*

3. Find the length and equations of the shortest distance between the lines

$$y=m_1x+c_1,\ z=a\,;$$
$$y=m_2x+c_2,\ z=-a.$$

Find also the locus of the middle point of the S D. if a is constant and m_1, c_1, m_2, c_2 vary. *(Gujarat, 1957)*

$$\left[\textbf{Ans.}\ 2a\,;\ \frac{x+\dfrac{c_1-c_2}{m_1-m_2}}{0}=\frac{y+\dfrac{m_2c_1-m_1c_2}{m_1-m_2}}{0}=\frac{z-a}{1}.\right]$$

4. A straight line whose direction cosines are l, m, n meets each of the lines

$$y=x\tan\alpha,\ z=c\,;$$
$$y=-x\tan\alpha,\ z=-c.$$

Find the shortest distance between this line and z-axis, and show that its equations are

$$lx+my=0,\ z(l^2+m^2)\sin\alpha\cos\alpha+clm=0.$$

(Bombay, 1959)

5. Find the equations to the line through the origin which meets at right angles the line whose equations are

$$(b+c)x+(c+a)y+(a+b)z=k=(b-c)x+(c-a)y+(a-b)z.$$

(Allahabad, 1959)

[**Ans.** $cx+ay+bz=0=lx+my+nz$, where $l=a^2-bc$, $m=b^2-ca$, $n=c^2-ab$.]

6. Show that if the axes are rectangular, the equations to the perpendicular from the point (x_1, y_1, z_1) to the plane

$$ax+by+cz+d=0$$

are
$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}.$$

(Punjab, 1962 ; Delhi Hons., 1953)

Hence find the foot and the length of the perpendicular from the point on the plane.

7. Prove that a line which passes through the point (α, β, γ) and intersects the parabola $y=0, z^2=4ax$, lies on the surface

$$(\beta z-\gamma y)^2=4a(\beta-y)(\beta x-\alpha y).$$ *(Vikram Engineering, 1960)*

8. Find the equations to the planes through the point $(-1, 0, 1)$ and the lines

$$4x-3y+1=0=y-4z+13\ ;$$
$$2x-y-2=0=z-5,$$

and show that the equations of the line through the given point which intersects the two given lines can be written as

$$x=y-1=z-2.$$ *(Raj. Engg., 1957 ; Punjab, 1958 S)*

9. From the point P (a, b, c) perpendiculars PA, PB are drawn to the lines

$$y=2x,\ z=1\ ;$$
$$y=-2x,\ z=-1.$$

Find the coordinates of A and B, and prove that if P moves so that angle APB is always a right angle, P lies on the surface

$$12x^2-3y^2+25z^2=25$$

(Nagpur T.D.C., 1962 ; Karnatak, 1956)

$$\left[\textbf{Ans.}\ \left(\frac{a+2b}{5}, \frac{2a+4b}{5}, 1\right);\ \left(\frac{a-2b}{5}, \frac{-2a+4b}{5}, 1\right).\right]$$

10. The axes being rectangular, find the equations to the perpendicular from the origin to the line

$$x+2y+3z+4=0,$$
$$2x+3y+4z+5=0.$$

Find also the coordinates of the foot of the perpendicular.

(Punjab, 1957 S ; Kashmir, 1957)

$$\left[\textbf{Ans.}\ \frac{x}{2}=\frac{y}{-1}=\frac{z}{-4}\ ;\ \left(\frac{2}{3}, -\frac{1}{3}, -\frac{4}{3}\right).\right]$$

11. Find the condition if three lines

$$\frac{x}{\alpha}=\frac{y}{\beta}=\frac{z}{\gamma},\quad \frac{x}{a\alpha}=\frac{y}{b\beta}=\frac{z}{c\gamma}\quad \text{and}\quad \frac{x}{l}=\frac{y}{m}=\frac{z}{n}$$

are to be coplanar. *(Punjab, 1960)*

$$\left[\textbf{Ans.}\ \frac{l}{\alpha}(b-c)+\frac{m}{\beta}(c-a)+\frac{n}{r}(a-b)=0.\right]$$

12. A line moves so as to intersect the line $z=0,\ x=y$; and the circles

$$x=0,\ y^2+z^2=r^2\ ;$$
$$y=0,\ z^2+x^2=r^2.$$

Prove that the equation to the locus is

$$(x+y)^2[z^2+(x-y)^2]=r^2(x-y)^2. \qquad \textit{(Delhi Hons., 1962)}$$

[**Hint.** *Let line be*

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad \ldots(1)$$

$\because$ *(1) intersects* $\dfrac{x}{1}=\dfrac{y}{1}=\dfrac{z}{0}$ $\qquad \ldots(2)$

$$\therefore \quad \begin{vmatrix} \alpha & \beta & \gamma \\ l & m & n \\ 1 & 1 & 0 \end{vmatrix}=0,$$

or,
$$-\alpha n+\beta n+\gamma\,(l-m)=0 \qquad \ldots(3)$$

$\because$ *(1) intersects* $x=0,\ y^2+z^2=r^2,$

$$\therefore \quad \left(\beta-\frac{m\alpha}{l}\right)^2+\left(\gamma-\frac{\alpha n}{l}\right)^2=r^2 \qquad \ldots(4)$$

$\because$ *(1) intersects* $y=0,\ z^2+x^2=r^2,$

$$\therefore \quad \left(\gamma-\frac{n\beta}{m}\right)^2+\left(\alpha-\frac{l\beta}{m}\right)^2=r^2 \qquad \ldots(5)$$

From (3),

$$(\alpha-\beta)n-\gamma\,(l-m)=0$$

$$\therefore \quad (\alpha-\beta)\,\frac{n}{m}=\gamma\left(\frac{l}{m}-1\right) \qquad \ldots(6)$$

Also,
$$(\alpha-\beta)\,\frac{n}{l}=\gamma\left(1-\frac{m}{l}\right) \qquad \ldots(7)$$

From (4) and (7),

$$\left(\beta-\frac{m\alpha}{\ }\right)^2+\left[\gamma-\frac{\alpha\gamma\left(1-\frac{m}{l}\right)}{\alpha-\beta}\right]^2=r^2 \qquad \ldots(8)$$

From (5) and (6),

$$\left[\gamma-\frac{\beta\gamma\left(\frac{l}{m}-\ \right)}{\alpha-\beta}\right]^2+\left(\alpha-\frac{l\beta}{m}\right)^2=r^2 \qquad \ldots(9)$$

From (8),

$$\left(\beta-\frac{m\alpha}{l}\right)^2+\gamma^2\left(\frac{\alpha m}{l}-\beta\right)^2\Big/(\alpha-\beta)^2=r^2,$$

or,
$$\left(\frac{m\alpha}{l}-\beta\right)^2[(\alpha-\beta)^2+\gamma^2]=(\alpha-\beta)^2\,r^2 \qquad \ldots(10)$$

From (9),

$$\gamma^2\left(\alpha-\frac{l\beta}{m}\right)^2\Big/(\alpha-\beta)^2+\left(\alpha-\frac{l\beta}{m}\right)^2=r^2,$$

or,
$$\left(-\alpha+\frac{\beta l}{m}\right)^2[\gamma^2+(\alpha-\beta)^2]=r^2\,(\alpha-\beta)^2 \qquad \ldots(11)$$

Divide (10) by (11), Hence etc.].

13. **Prove that**

$$\frac{a}{y-z}+\frac{b}{z-x}+\frac{c}{x-y}=0$$

represents a pair of planes whose line of intersection is equally inclined to the axes. **(Baroda, 1954)**

14. The ends of the diameters of the ellipse $z=c,\ \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

are joined to the corresponding ends of the conjugates of parallel diameters of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z=-c.$$

Find the equation to the surface generated by the joining lines. **(Agra, 1950)**

$$\left[\textbf{Ans.}\quad \frac{2x^2}{a^2}+\frac{2y^2}{b^2}-\frac{z^2}{c^2}=1.\right]$$

[**Hint.** *Let the coordinates of the ends P and D of the semi-conjugate diameters CP and CD of the ellipse*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z=c$$

is $(a\cos\theta,\ b\sin\theta,\ c)$

and $(-a\sin\theta,\ b\cos\theta,\ c)$ *respectively.*

The corresponding ends of the parallel conjugate diameter of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z=-c$$

are given by $P_1\ (a\cos\theta,\ b\sin\theta,\ -c)$ *and* $D_1\ (-a\sin\theta,\ b\cos\theta,\ -c)$.

The equations of PD_1 *are*

$$\frac{x-a\cos\theta}{a(\cos\theta+\sin\theta)}=\frac{y-b\sin\theta}{b(\sin\theta-\cos\theta)}=\frac{z-c}{2c} \qquad \dots(1)^*$$

$$\therefore \qquad (z+c)\cos\theta+(z-c)\sin\theta=\frac{2cx}{a}$$

and $$(z+c)\sin\theta-(z-c)\cos\theta=\frac{2cy}{b}.$$

Squaring and adding, we have etc.]

**We can also eliminate θ as follows :*

From (1), taking the first and third relations, we have

$$\frac{x}{a}=\frac{z+c}{2c}\cos\theta+\frac{z-c}{2c}\sin\theta.$$

From (1), taking the second and third relations, we have

$$\frac{y}{b}=\frac{z+c}{2c}\sin\theta-\frac{z-c}{2c}\cos\theta.\quad \textit{Square and add.}$$

15. Find the equation to the surface generated by a straight line which is parallel to the plane $z=0$ and intersects the line $x=y=z$ and the curve

$$x+2y=4z,\ x^2+y^2=a^2.$$ *(Allahabad, 1963)*

[**Ans.** $z^2(x+3y-4z)^2+4z^2(x+y-2z)^2=a^2(x+2y-3z)^2$.]

16. *If the axes are rectangular, and if l_1, m_1, n_1 ; l_2, m_2, n_2 are direction cosines, show that the equations to the planes through the lines which bisect the angles between*

$$\frac{x}{l_1}=\frac{y}{m_1}=\frac{z}{n_1}$$

and

$$\frac{x}{l_2}=\frac{y}{m_2}=\frac{z}{n_2},$$

and at right angles to the plane containing them, are

$$(l_1 \pm l_2)x+(m_1 \pm m_2)y+(n_1 \pm n_2)z=0.$$

Sol. Let $P(x, y, z)$ be **any** point on the required plane passing through the bisectors of the given lines

$$\frac{x}{l_1}=\frac{y}{m_1}=\frac{z}{n_1} \quad ...(1),$$

$$\frac{x}{l_2}=\frac{y}{m_2}=\frac{z}{n_2} \quad ...(2)$$

and perpendicular to the plane containing (1) and (2).

Now, the direction ratios of OP are x, y, z, where O is the origin.

$\therefore$ the direction cosines of OP are

$$\frac{x}{\sqrt{x^2+y^2+z^2}},\ \frac{y}{\sqrt{x^2+y^2+z^2}},\ \frac{z}{\sqrt{x^2+y^2+z^2}}.$$

Now OP makes equal angles with (1) and (2).

Let α be this angle.

$$\therefore \quad \frac{l_1x+m_1y+n_1z}{\sqrt{x^2+y^2+z^2}}=\mp\frac{l_2x+m_2y+n_2z}{\sqrt{x^2+y^2+z^2}},$$

or, $$(l_1x+m_1y+n_1z)=\pm(l_2x+m_2y+n_2z),$$

or, $$(l_1 \pm l_2)x+(m_1 \pm m_2)y+(n_1 \pm n_2)z=0.$$

Aliter. Ttho direction ratios of the bisectors of the angles between the given lines are

$l_1 \pm l_2,\ m_1 \pm m_2,\ n_1 \pm n_2$. (The students are advised to find them.)

From Geometry, the external and internal bisectors of the angles between two lines are always at right angles to each other.

$\therefore$ one of the bisectors will be a normal to a plane containing the other, provided it is perpendicular to the plane containing those lines.

$\therefore$ the required planes are those planes whose normals are the bisecting lines.

$\therefore$ their equations are

$$(l_1 \pm l_2)x+(m_1 \pm m_2)y+(n_1 \pm n_2)z=0.$$

17. The lengths of two opposite edges of a tetrahedron are a, b, their S.D. is equal to d, and the angle between them is θ; prove that the volume is $\frac{abd \sin \theta}{6}$.

(Agra, 1962; Raj., 1954; Banaras, 1962; Bihar, 1962; Calcutta 1964)

Sol. Let OABC be tetrahedron in which OA and BC are opposite edges of lengths a and b respectively.

Let O be taken as the origin.

Let the coordinates of A, B, C referred to O as origin and rectangular axes through O as the axes of coordinates be (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) respectively.

Let l, m, n be direction cosines of OA.

$$\therefore \quad x_1 = la, \; y_1 = ma, \; z_1 = na,$$

or,
$$l = \frac{x_1}{a}, \; m = \frac{y_1}{a}, \; n = \frac{z_1}{a}.$$

$\therefore$ equations of OA are $\dfrac{x}{x_1/a} = \dfrac{y}{y_1/a} = \dfrac{z}{z_1/a}$.

Equations to BC are $\dfrac{x - x_2}{(x_2 - x_3)/b} = \dfrac{y - y_2}{(y_2 - y_3)/b} = \dfrac{z - z_2}{(z_2 - z_3)/b}$.

$\therefore$ shortest distance between OA and BC is

$$d = \begin{vmatrix} x_2 & y_2 & z_2 \\ (x_2 - x_3)/b & (y_2 - y_3)/b & (z_2 - z_3)/b \\ x_1/a & y_1/a & z_1/a \end{vmatrix} \div \sin \theta,$$

where θ is the angle between OA and BC,

or,
$$d \sin \theta = \frac{1}{ab} \begin{vmatrix} x_2 & y_2 & z_2 \\ -x_3 & -y_3 & -z_3 \\ x_1 & y_1 & z_1 \end{vmatrix}$$

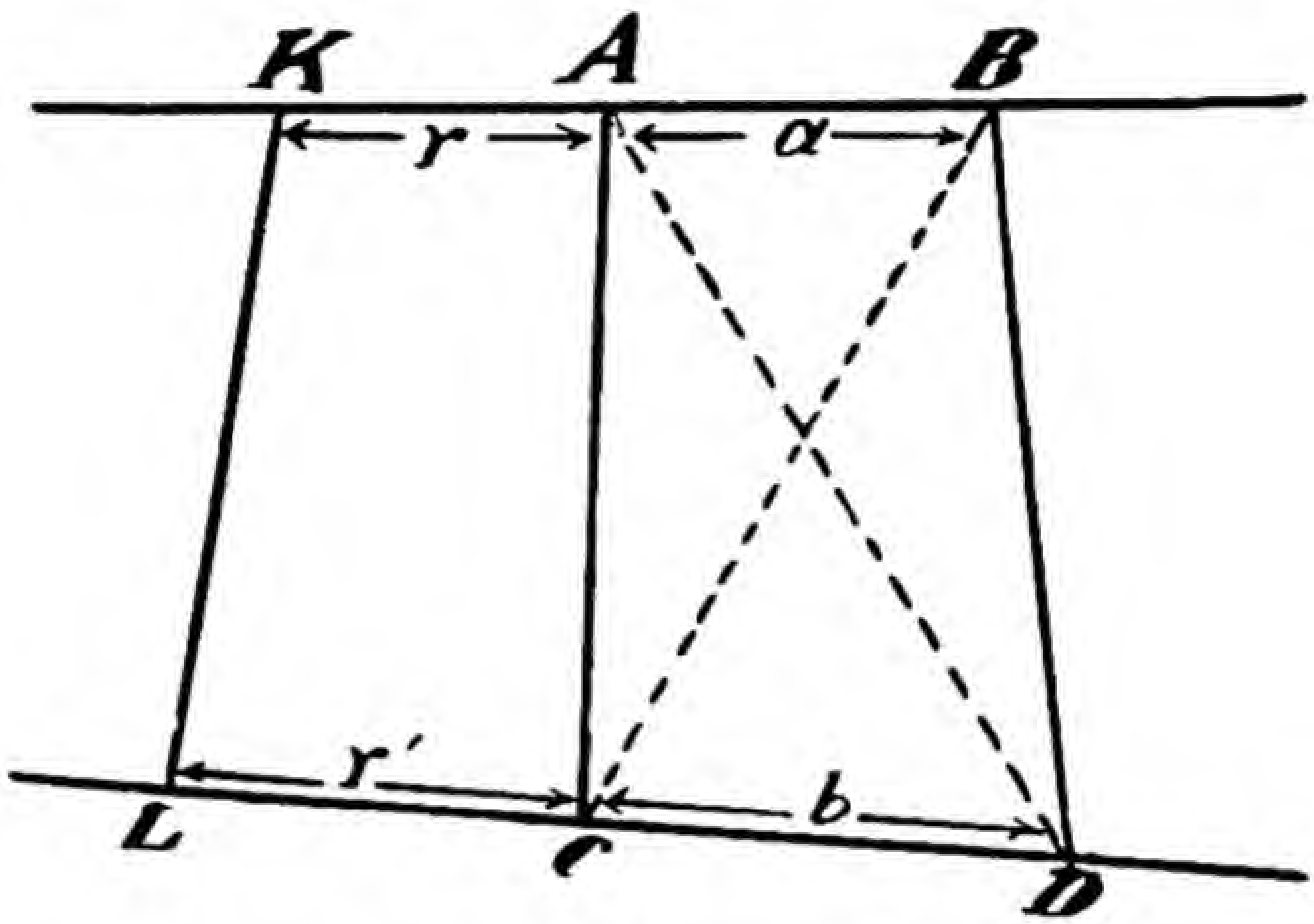
$$= -\frac{1}{ab} \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \qquad \ldots(1)$$

Now, volume of the tetrahedron OABC is

$$= \frac{1}{6} \begin{vmatrix} 0 & 0 & 0 & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix}$$

$$=-\frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}=\frac{1}{6}\ abd \sin\theta, \text{ using (1)}$$

Aliter. Let the equations of the opposite edges



AB($=a$) and CD($=b$) be $y=mx,\ z=c$

and $y=-mx,\ z=-c,$

or, $y=x\tan\alpha,\ z=c : y=-x\tan\alpha,\ z=-c,$

$$\text{or, } \frac{x}{\cos\alpha}=\frac{y}{\sin\alpha}=\frac{z-c}{0} \qquad \ldots(1)$$

$$\text{and } \frac{x}{\cos\alpha}=\frac{y}{-\sin\alpha}=\frac{z+c}{0} \qquad \ldots(2)$$

Putting each member of (1) equal to r, the coordinates of A are ($r\cos\alpha$, $r\sin\alpha$, c).

Putting each member of (1) equal to $r+a$, the coordinates of B are

$[(r+a)\cos\alpha,\ (r+a)\sin\alpha,\ c]$

Putting each member of (2) equal to r', the coordinates of C are

$(r'\cos\alpha,\ -r'\sin\alpha,\ -c)$.

Putting each member of (2) equal to $(r'+b)$, the coordinates of D are

$[(r'+b)\cos\alpha,\ -(r'+b)\sin\alpha,\ -c]$.

$\therefore$ volume of the tetrahedron A, BCD

$$=\frac{1}{6}\begin{vmatrix} r\cos\alpha & r\sin\alpha & c & 1 \\ (r+a)\cos\alpha & (r+a)\sin\alpha & c & 1 \\ r'\cos\alpha & -r'\sin\alpha & -c & 1 \\ (r'+b)\cos\alpha & -(r'+b)\sin\alpha & -c & 1 \end{vmatrix}$$

$$=\frac{1}{6}\begin{vmatrix} -a\cos\alpha & -a\sin\alpha & 0 & 0 \\ (r+a-r')\cos\alpha & (r+a+r')\sin\alpha & 2c & 0 \\ -b\cos\alpha & b\sin\alpha & 0 & 0 \\ (r'+b)\cos\alpha & -(r'+b)\sin\alpha & -c & 1 \end{vmatrix}$$

on subtracting second row from the first, third row from the second and fourth row from the third row,

$$=\frac{1}{6}\begin{vmatrix} -a\cos\alpha & -a\sin\alpha & 0 \\ (r+a-r')\cos\alpha & (r+a+r')\sin\alpha & 2c \\ -b\cos\alpha & b\sin\alpha & 0 \end{vmatrix}$$

$$=\frac{1}{6}[-a\cos\alpha\{-2bc\sin\alpha\}+a\sin\alpha\,\{2bc\cos\alpha\}]$$

$$=\frac{1}{6}[2abc\sin\alpha\cos\alpha+2abc\sin\alpha\cos\alpha]$$

$$=\frac{1}{6}abc\sin 2\alpha$$

$$=\frac{1}{6}\,abd\sin\theta,\qquad \text{since}\quad 2c=d \quad\text{and}\quad 2\alpha=\theta \text{ (given)}.$$

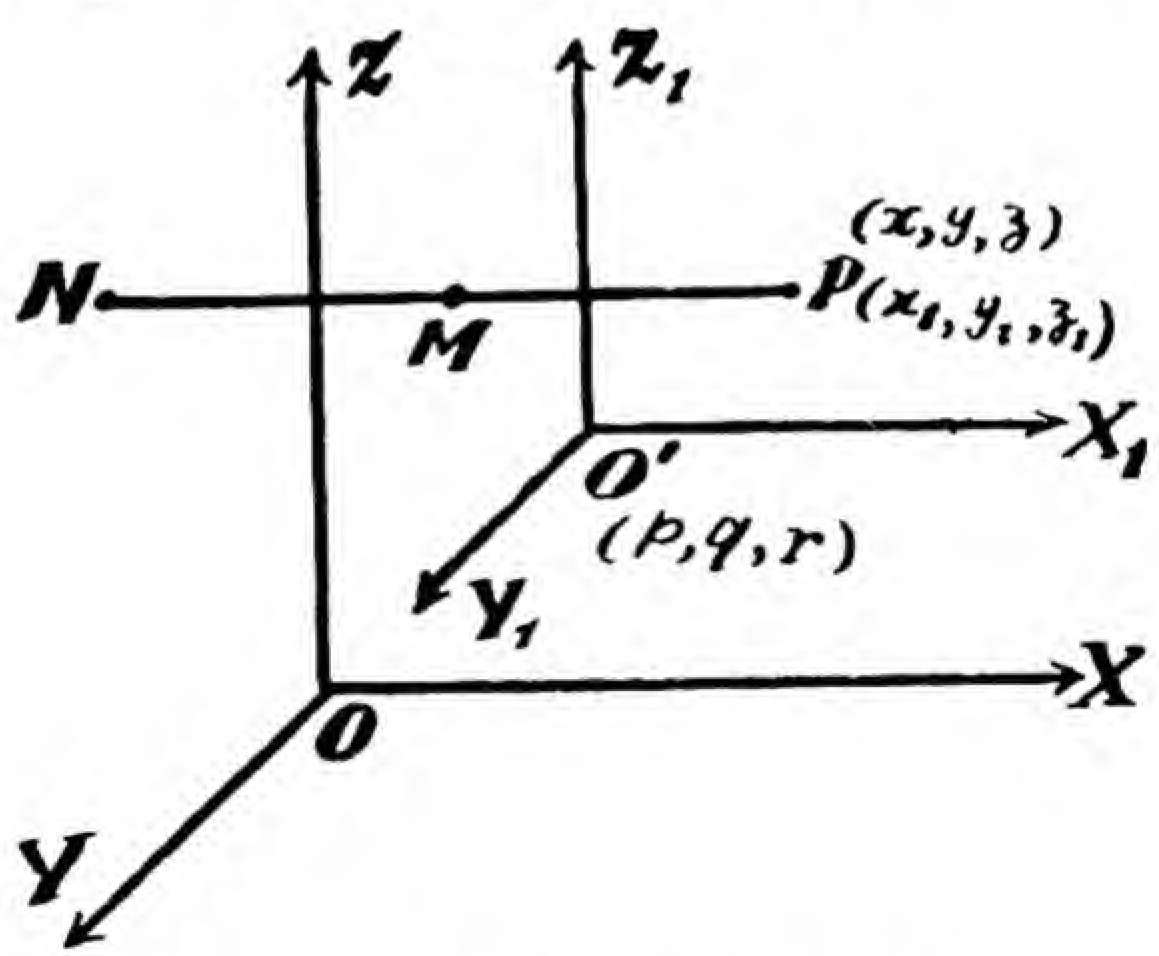
5

Change of Axes and Transformation of Coordinates

5·1. Change of origin.

To change the origin of coordinates without changing the directions of the coordinate axes.

Let OX, OY, OZ be the old axes and $O'X_1$, $O'Y_1$ and $O'Z_1$ be the new axes respectively parallel to the old axes.



Let the coordinates of O′ referred to old axes be (p, q, r).

Let the coordinates of **any** point p be (x, y, z) referred to the old axes and (x_1, y_1, z_1) referred to the new axes.

Draw PN perpendicular to YOZ plane cutting $Y_1O'Z_1$ plane in M.

$\therefore$ NP$=x$ and MP$=x_1$.

Now, NM is the length of the perpendicular from O′ on the YOZ plane and is therefore equal to p.

Also, $\quad NP = NM + MP$,

or, $\quad \mathbf{x} = \mathbf{x}_1 + \mathbf{p}$.

Similarly, $\quad \mathbf{y} = \mathbf{y}_1 + \mathbf{q} \quad$ and $\quad \mathbf{z} = \mathbf{z}_1 + \mathbf{r}$.

Hence $\mathbf{x} = \mathbf{x}_1 + \mathbf{p}$, $\mathbf{y} = \mathbf{y}_1 + \mathbf{q}$, $\mathbf{z} = \mathbf{z}_1 + \mathbf{r}$ are the transformation formulae from the old to the new axes.

From above, $x_1 = x - p$, $y_1 = y - q$ and $z_1 = z - r$.

Hence $\mathbf{x}_1 = \mathbf{x} - \mathbf{p}$, $\mathbf{y}_1 = \mathbf{y} - \mathbf{q}$, $\mathbf{z}_1 = \mathbf{z} - \mathbf{r}$ are the transformation formulae from the new to the old axes.

Note 1. Application of the above transformation formulae.

(A) To find the coordinates of a point referred to the new axes, given its coordinates referred to the old axes.

If the coordinates of a point be (x, y, z) referred to the old axes, we can find the coordinates of the same point referred to the new axes by subtracting the x-coordinate of the new origin from the x co ordinate of the given point, y coordinate of the new origin from the y co-ordinate of the given point and z coordinate of the new origin from the z-co-ordinate of the given point, all referred to the old axes.

$\therefore$ *if (p, q, r) be the coordinates of the new origin referred to the old axes, the coordinates (x, y, z) of the given point referred to* **new** *axes are $(x-p, y-q, z-r)$.*

(B) To find the equation of a surface referred to the new axes, given its equation referred to the old axes.

If the equation of a surface referred to the **old axes** *be $f(x, y, z) = 0$, its equation referred to the* **new axes** *will be obtained by writing $\mathbf{x}_1 + \mathbf{p}$ for $\mathbf{x}$, $\mathbf{y}_1 + \mathbf{q}$ for $\mathbf{y}$ and $\mathbf{z}_1 + \mathbf{r}$ for $\mathbf{z}$ in the given equation, x_1, y, z_1 being taken as the current coordinates referred to the new axes.*

An advice. *Since it is usually convenient to denote the current coordinates of a point by the letters x, y, z, therefore it is suggested that suffixes may be dropped in the transformed equation, remembering that the equation is referred to the* **new** *axes.*

Hence the equation of a surface referred to the new axes may be obtained by simply writing $\mathbf{x} + \mathbf{p}$(not $\mathbf{x}_1 + \mathbf{p}$) for $\mathbf{x}$,

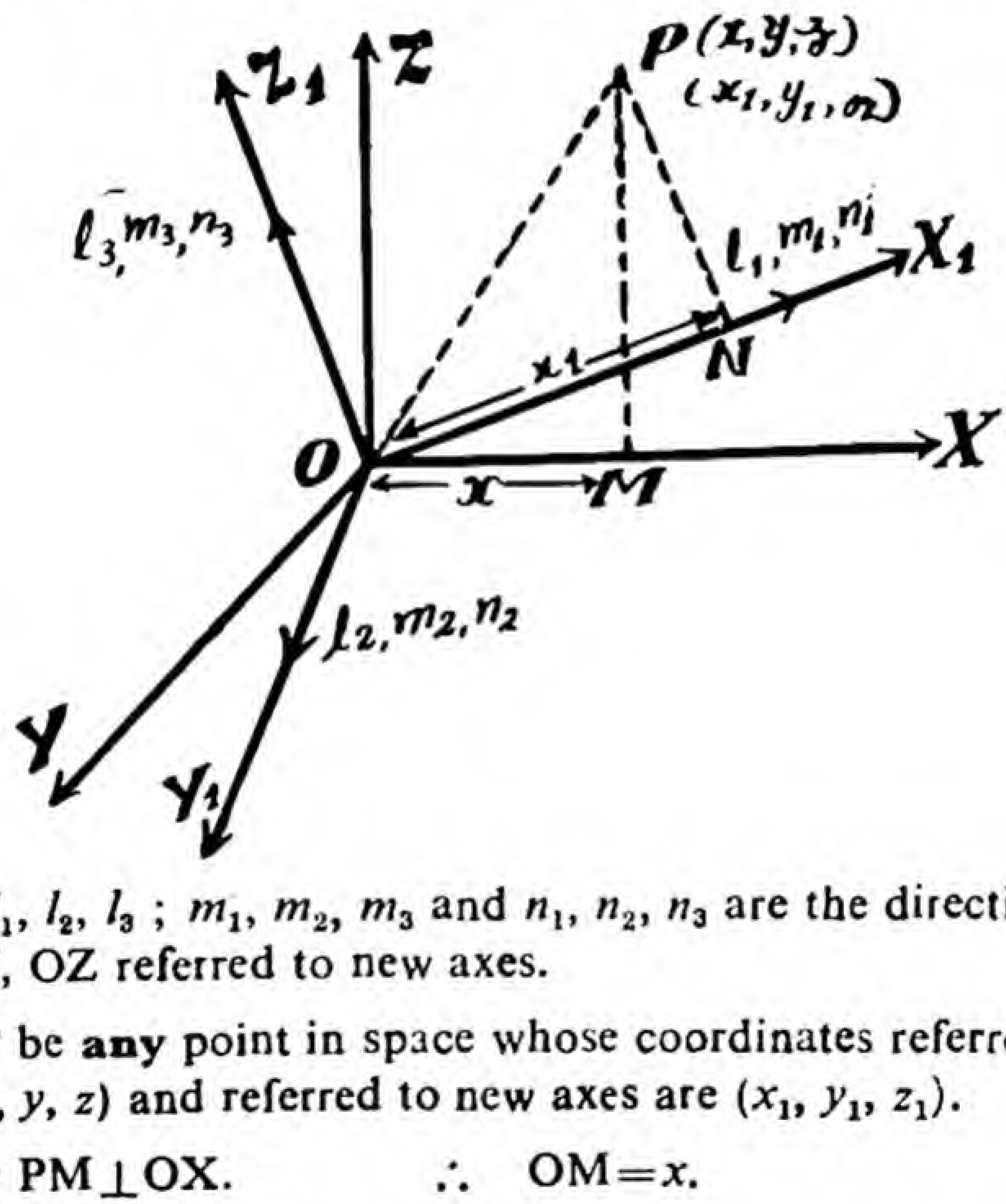
$y+q$ (not y_1+q) for y and $z+r$ (not z_1+r) for z in the equation of the surface referred to the old axes.

Note 2. Coordinates of the old origin referred to the new axes.

The coordinates of the old origin referred to the new axes are $(-p, -q, -r)$.

5.2. Change of direction of axes without change of origin. To change the directions of the axes without changing the origin. *(Punjab B.Sc., 1963)*

Let OX, OY, OZ be the original axes and OX_1, OY_1, OZ_1 be the new axes. Let l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3 be the direction cosines of OX_1, OY_1, OZ_1 respectively referred to the old axes.



$\therefore$ l_1, l_2, l_3 ; m_1, m_2, m_3 and n_1, n_2, n_3 are the direction cosines of OX, OY, OZ referred to new axes.

Let P be **any** point in space whose coordinates referred to old axes are (x, y, z) and referred to new axes are (x_1, y_1, z_1).

Draw PM $\perp$ OX. $\quad\therefore$ OM $=x$.

Now, OM = projection of OP

(the line joining the points $(0, 0, 0)$, (x_1, y_1, z_1)

on OX (the line whose direction cosines are l_1, l_2, l_3),

or. $\quad x=(x_1-0)l_1+(y_1-0)l_2+(z_1-0)l_3$.

$$\left.\begin{aligned} \therefore\qquad & \mathbf{x=x_1l_1+y_1l_2+z_1l_3} \\ \text{Similarly,}\quad & \mathbf{y=x_1m_1+y_1m_2+z_1m_3} \\ & \mathbf{z=x_1n_1+y_1n_2+z_1n_3} \end{aligned}\right\} \quad \dots\mathbf{(A)}$$

Further, let PN be perpendicular from P on OX_1.

$\therefore$ $ON = x_1$.

Now, ON = projection of OP (the line joining the points (0, 0, 0,) and (x, y, z) on the line OX_1 the line whose direction cosines are l_1, m_1, n_1)

or $$x_1 = (x-0)\, l_1+(y-0)m_1+(z-0)n_1$$

$\therefore$
Similarly
$$\left.\begin{aligned} \mathbf{x_1} &= \mathbf{xl_1+ym_1+zn_1} \\ \mathbf{y_1} &= \mathbf{xl_2+ym_2+zn_2} \\ \mathbf{z_1} &= \mathbf{xl_3+ym_3+zn_3} \end{aligned}\right\} \quad \ldots\textbf{(B)}$$

(A) and (B) are the required transformation formulae.

Aid to memory. The above transformation formulae can be remembered from the following table :

(1) *Write down in four columns x, y, z ; l_1, m_1, n_1 ; l_2, m_2, n_2 and l_3, m_3, n_3. Leave a blank space at the head of the first column and write x_1, y_1, z_1 at the heads of the other columns.*

	x_1	y_1	z_1
x	l_1	l_2	l_3
y	m_1	m_2	m_3
z	n_1	n_2	n_3

(2) **To get x, multiply the numbers in x-row by the number at the heads of their respective columns and add.**

Thus $\mathbf{x=l_1x_1+l_2y_1+l_3z_1}$.

Similarly, obtain y snd z.

(3) **To get $\mathbf{x_1}$, multiply the numbers in $\mathbf{x_1}$-column by the numbers at extreme left of their respective rows and add. Thus $\mathbf{x_1=l_1x+m_1y+n_1z}$. Similarly, obtain $\mathbf{y_1}$ and $\mathbf{z_1}$.**

5·3. **To show that the degree of an equation remains unchanged by any transformation of axes.**

Proof. From Art. 5·1, if the origin be changed to (p, q, r) with axes remaining parallel to their original directions, then x is changed to $x+p$, y to $y+q$ and z to $z+r$. ...(1)

From Art. 5·2, if the directions of the axes are changed to lines having direction cosines l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3, without

changing the origin, x is changed to $l_1x+l_2y+l_3z$, y to $m_1x+m_2y+m_3z$, z to $n_1x+n_2y+n_3z$. ...(2)

From (1) and (2), in whatever manner the axes are changed, x is changed to $l_1x+l_2y+l_3z+p$, y to $m_1x+m_2y+m_3z+q$ and z to $n_1x+n_2y+n_3z+r$.

These are expressions of the first degree in x, y and z.

$\therefore$ the degree of the given equation cannot increase. ...(3)

Also, the degree of the equation cannot decrease, for otherwise on retransforming, it must increase, which is impossible from (3). Hence the degree of the equation is unaltered by any transformation of axes.

This proves the proposition.

5·4. Relations between the direction cosines of three mutually perpendicular lines.

To find the relations between the direction cosines of three mutually perpendicular lines whose direction cosines are l_1, m_1, n_1; l_2, m_2, n_2 and l_3, m_3, n_3.

Let l_1, m_1, n_1; l_2, m_2, n_2 and l_3, m_3, n_3 be the direction cosines of three mutually perpendicular lines OX_1, OY_1 and OZ_1.

$\because$ l_1, m_1, n_1; l_2, m_2, n_2 and l_3, m_3, n_3 are actual direction cosines,

$$\therefore \quad \left.\begin{aligned} l_1^2+m_1^2+n_1^2&=1\\ l_2^2+m_2^2+n_2^2&=1\\ l_3^2+m_3^2+n_3^2&=1 \end{aligned}\right\} \quad \text{..(A)}$$

$\because$ the lines are mutually perpendicular,

$$\therefore \quad \left.\begin{aligned} l_1l_2+m_1m_2+n_1n_2&=0\\ l_1l_3+m_1m_3+n_1n_3&=0\\ l_2l_3+m_2m_3+n_2n_3&=0 \end{aligned}\right\} \quad \text{...(B)}$$

Again, l_1, l_2, l_3; m_1, m_2, m_3 and n_1, n_2, n_3 are the direction—cosines of the original axes OX, OY, OZ referred to new axes OX_1, OY_1, OZ_1.

$$\therefore \quad \left.\begin{aligned} l_1^2+l_2^2+l_3^2&=1\\ m_1^2+m_2^2+m_3^2&=1\\ n_1^2+n_2^2+n_3^2&=1 \end{aligned}\right\} \quad \text{...(C)}$$

$$\left.\begin{array}{l} \mathbf{l_1m_1 + l_2m_2 + l_3m_3 = 0} \\ \mathbf{l_1n_1 + l_2n_2 + l_3n_3 = 0} \\ \mathbf{m_1n_1 + m_2n_2 + m_3n_3 = 0} \end{array}\right\} \quad \ldots\textbf{(D)}$$

Hence (**A**), (**B**), (**C**) **and** (**D**) are the required relations.

Note. *The above relations are not independent. In fact (C) and (D) can be obtained algebraically from (A) and (B).*

5·5. If $\mathbf{l_1, m_1, n_1}$; $\mathbf{l_2, m_2, n_2}$, and $\mathbf{l_3, m_3, n_3}$ be the direction consines of three mutually perpendicular lines, to prove that

(i) $$\begin{vmatrix} \mathbf{l_1} & \mathbf{m_1} & \mathbf{n_1} \\ \mathbf{l_2} & \mathbf{m_2} & \mathbf{n_2} \\ \mathbf{l_3} & \mathbf{m_3} & \mathbf{n_3} \end{vmatrix} = \pm 1,$$

(ii) $$\mathbf{l_1 = \pm (m_2n_3 - m_3n_2)}$$
$$\mathbf{m_1 = \pm (n_2l_3 - n_3l_2)}$$
$$\mathbf{n_1 = \pm (l_2m_3 - l_3m_2)}.$$

Proof.

Let $\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$ be denoted by $\triangle$.

(*i*) $$\triangle^2 = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$

$$= \begin{vmatrix} l_1^2 + m_1^2 + n_1^2 & l_1l_2 + m_1m_2 + n_1n_2 & l_1l_3 + m_1m_3 + n_1n_3 \\ l_2l_1 + m_2m_1 + n_2n_1 & l_2^2 + m_2^2 + n_2^2 & l_2l_3 + m_2m_3 + n_2n_3 \\ l_1l_3 + m_1m_3 + n_1n_3 & l_3l_2 + m_3m_2 + n_3n_2 & l_3^2 + m_3^2 + n_3^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1, \text{ using Art. 5·4.}$$

$\therefore\quad \Delta=\pm 1.$

(ii) $\because\quad l_1 l_2+m_1 m_2+n_1 n_2=0$

$l_1 l_3+m_1 m_3+n_1 n_3=0,$

$$\therefore\quad \frac{l_1}{m_2 n_3-m_3 n_2}=\frac{m_1}{n_2 l_3-n_3 l_2}=\frac{n_1}{l_2 m_3-l_3 m_2}$$

$$=\frac{\sqrt{l_1^2+m_1^2+n_1^2}}{\pm\sqrt{\Sigma(m_2 n_3-m_3 n_2)^2}}$$

$$=\frac{1}{\pm\sin 90^\circ},$$ because the angle between the lines is 90°.

$$=\pm 1.$$

$$\therefore\quad l_1=\pm(m_2 n_3-m_3 n_2),\ m_1=\pm(n_2 l_3-n_3 l_2),$$
$$n_1=\pm(l_2 m_3-l_3 m_2).$$

This proves the proposition.

Note. *In* Δ, *each element* $=\pm$(*its cofactor*).

EXAMPLES V

Type I. Ex 1. Find the equation of the surface

$$3x^2+5y^2+3z^2+2yz+2xz+2xy=1,$$

with reference to axes through the same origin with direction cosines proportional to $-1, 0, 1$; $1, -1, 1$ and $1, 2, 1$.

Sol. The given equation is

$$3x^2+5y^2+3z^2+2yz+2zx+2xy-1=0 \qquad \ldots(1)$$

Here $l_1=\frac{-1}{\sqrt{2}},\ m_1=0,\ n_1=\frac{1}{\sqrt{2}}\ ;\ l_2=\frac{1}{\sqrt{3}},$

$m_2=\frac{-1}{\sqrt{3}},\ n_2=\frac{1}{\sqrt{3}}\ ;\quad l_3=\frac{1}{\sqrt{6}},\quad m_3=\frac{2}{\sqrt{6}},\ n_3=\frac{1}{\sqrt{6}}.$

$\therefore$ transformed equation is

$$3\left(-\frac{x}{\sqrt{2}}+\frac{y}{\sqrt{3}}+\frac{z}{\sqrt{6}}\right)^2+5\left(0.x-\frac{1}{\sqrt{3}}.y+\frac{2z}{\sqrt{6}}\right)^2$$
$$+3\left(\frac{x}{\sqrt{2}}+\frac{1}{\sqrt{3}}y+\frac{1}{\sqrt{6}}z\right)^2+2\left(0.x-\frac{1}{\sqrt{3}}\ y+\frac{2}{\sqrt{6}}z\right)\left(\frac{x}{\sqrt{2}}+\frac{y}{\sqrt{3}}+\frac{1}{\sqrt{6}}z\right)$$
$$+2\left(\frac{1}{\sqrt{2}}x+\frac{1}{\sqrt{3}}y+\frac{1}{\sqrt{6}}z\right)\left(-\frac{x}{\sqrt{2}}+\frac{y}{\sqrt{3}}+\frac{z}{\sqrt{6}}\right)$$
$$+2\left(-\frac{x}{\sqrt{2}}+\frac{y}{\sqrt{3}}+\frac{z}{\sqrt{6}}\right)\left(0.x-\frac{1}{\sqrt{3}}.y+\frac{2}{\sqrt{6}}z\right)-1=0,$$

or,
$$\frac{3}{6}\left(-\sqrt{3}x+\sqrt{2}y+z\right)^2+\frac{5}{6}\left(2z-\sqrt{2}y\right)^2$$
$$+\frac{3}{6}\left(\sqrt{3}x+\sqrt{2}y+z\right)^2+\frac{2}{6}\left(2z-\sqrt{2}y\right)\left(\sqrt{3}x+\sqrt{2}y+z\right)$$
$$+\frac{2}{6}\left(\sqrt{3}.x+\sqrt{2}y+z\right).\left(-\sqrt{3}.x+\sqrt{2}.y+z\right)$$
$$+\frac{2}{6}\left(-\sqrt{3}.x+\sqrt{2}.y+z\right)\left(2z-\sqrt{2}y\right)=1.$$

or,
$$3[3x^2+2y^2+z^2-2\sqrt{6}xy-2\sqrt{3}xz+2\sqrt{2}yz]$$
$$+5[4z^2+2y^2-4\sqrt{2}yz]+3[3x^2+2y^2+z^2+2\sqrt{6}xy+2\sqrt{2}yz+2\sqrt{3}xz]$$
$$+2[2\sqrt{3}xz-\sqrt{6}xy+2\sqrt{2}yz-2y^2+2z^2-\sqrt{2}yz]$$
$$+2[-3x^2-\sqrt{6}xy-\sqrt{3}xz+\sqrt{6}xy+2y^2+\sqrt{2}yz+\sqrt{3}xz+\sqrt{2}yz+z^2]$$
$$+2[-2\sqrt{3}xz+2\sqrt{2}yz+2z^2+\sqrt{6}xy-2y^2-\sqrt{2}yz]=6,$$

or,
$$2x^2+3y^2+6z^2=1.$$

Ex. 2. What does the equation
$$x^2+7y^2+z^2+8yz+16zx-8xy=9$$
become when the lines joining the origin to the points $(1, 2, 2)$, $(2, -2, 1)$, $(2, 1, -2)$ are taken as the axes ? [**Ans.** $x^2+y^2-z^2=1$.]

Type II. Ex. 1. OA, OB, OC are three mutually perpendicular lines through the origin, and their direction cosines are
$$\mathbf{l_1, m_1, n_1\ ;\ \ l_2, m_2, n_2\ ;\ \ l_3, m_3, n_3.}$$
If OA = OB = OC = a, prove that equation of the plane ABC is
$$\mathbf{(l_1+l_2+l_3)x+(m_1+m_2+m_3)y+(n_1+n_2+n_3)z=0}$$ *(Punjab, 1958)*

Sol. Let $lx+my+nz=p$......(1) be the required equation.

The coordinates of A, B, C are respectively

(al_1, am_1, an_1), (al_2, am_2, an_2), (al_3, am_3, an_3).

$\because$ (1) passes through A, B, C,

$$\therefore\ a(ll_1+mm_1+nn_1)+p=0 \quad ...(2)$$
$$a(ll_2+mm_2+nn_2)+p=0 \quad ..(3)$$
$$a(ll_3+mm_3+nn_3)+p=0 \quad ...(4)$$

Multiplying (2) by l_1, (3) by l_2 and (4) by l_3, and adding, we have
$$al(l_1^2+l_2^2+l_3^2)+am(l_1m_1+l_2m_2+l_3m_3)$$
$$+an(l_1n_1+l_2n_2+l_3n_3)+p(l_1+l_2+l_3)=0,$$

or, $al=-p(l_1+l_2+l_3)$, using Art. 5.4.

or
$$\frac{l}{p}=-\frac{l_1+l_2+l_3}{a}.$$

Similarly, $\dfrac{m}{p}=-\dfrac{m_1+m_2+m_3}{p}$, $\dfrac{n}{p}=-\dfrac{n_1+n_2+n_3}{a}$.

$\therefore$ (1) becomes $(l_1+l_2+l_3)x+(m_1+m_2+m_3)y+(n_1+n_2+n_3)z=a$.

Ex. 2. The equations referred to rectangular axes of three mutually perpendicular planes are
$$p_r-l_rx-m_ry-n_rz=0,\ r=1, 2, 3.$$
Prove that if (α, β, γ) be at a distance d from each of them, then

$$d=\frac{\alpha-(l_1p_1+l_2p_2+l_3p_3)}{l_1+l_2+l_3}$$
$$=\frac{\beta-(m_1p_1+m_2p_2+m_3p_3)}{m_1+m_2+m_3}=\frac{\gamma-(n_1p_1+n_2p_2+n_3p_3)}{n_1+n_2+n_3}.$$

Ex. 3. If l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3 are the direction cosines of three mutually perpendicular lines, prove that the line whose direction cosines are proportional to $l_1+l_2+l_3$, $m_1+m_2+m_3$ $n_1+n_2+n_3$ makes equal angles with them.

MISCELLANEOUS (REVISION) EXAMPLES ON CHAPTER V

1. If three rectangular axes be rotated about the line $\frac{x}{\lambda}=\frac{y}{\mu}=\frac{z}{\nu}$ into new positions and the direction cosines of the new axes referred to the old are l_1, m_1, n_1, etc , then if $l_1=(m_2n_3-m_3n_2)$, show that

$$\lambda(m_3+n_2)=\mu(n_1+l_3)=\nu(l_2+m_1).$$

2. If (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3) be the direction-cosines of lines OX′, OY′, OZ′ referred to OX, OY, OZ as axes and θ_1, θ_2 the angles which the projections of OX′, OY′ on the plane XOY make with OX, then show that

$$\tan(\theta_1-\theta_2)=\pm\frac{n_3}{n_1n_2}.$$

3. Show that if $ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d$ be transformed by change of coordinates from one set of rectangular axes to another with the same origin, the expression $a+b+c$, $u^2+v^2+w^2$ remain unaltered in value. *(Punjab, 1962 ; Raj., 1954)*

Sol. Let l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3 be the direction cosines of the new axes referred to the old axes.

$\therefore$ putting $l_1x+l_2y+l_3z$ for x, $m_1x+m_2y+m_3z$ for y

and $n_1x+n_2y+n_3z$ for z in the given expression,

we have
$$a(l_1x+l_2y+l_3z)^2+b(m_1x+m_2y+m_3z)^2$$
$$+c(n_1x+n_2y+n_3z)^2+2f(m_1x+m_2y+m_3z)(n_1x+n_2y+n_3z)$$
$$+2g(n_1x+n_2y+n_3z)(l_1x+l_2y+l_3z)$$
$$+2h(l_1x+l_2y+l_3z)(m_1x+m_2y+m_3z)$$
$$+2u(l_1x+l_2y+l_3z)+2v(m_1x+m_2y+m_3z)$$
$$+2w(n_1x+n_2y+n_3z)+d,$$

or, $a_1x^2+b_1y^2+c_1z^2+2f_1yz+2g_1zx+2h_1xy+2u_1x$
$+2v_1y+2w_1z+d_1$, where

$$a_1=al_1^2+bm_1^2+cn_1^2+2fm_1n_1+2gn_1l_1+2hl_1m_1,$$
$$b_1=al_2^2+bm_2^2+cn_2^2+2fm_2n_2+2gn_2l_2+2hl_2m_2,$$
$$c_1=al_3^2+bm_3^2+cn_3^2+2fm_3n_3+2gn_3l_3+2hl_3m_3,$$
$$u_1=ul_1+vm_1+wn_1,$$
$$v_1=ul_2+vm_2+wn_2,$$
$$w_1=ul_3+vm_3+wn_3,$$

and $d_1=d$.

Now, $a_1+b_1+c_1=a\Sigma l_1^2+b\Sigma m_1^2+c\Sigma n_1^2+2f\Sigma m_1n_1+2g\Sigma n_1l_1$
$+2h\Sigma l_1m_1$
$=a+b+c$, because $\Sigma l_1^2=1$, etc.
and $\Sigma m_1n_1=0$, etc. **(Art. 5·4)**

Also, $u_1^2+v_1^2+w_1^2=u^2\Sigma l_1^2+v^2\Sigma m_1^2+w^2\Sigma n_1^2$
$+2uv\Sigma l_1m_1+2uw\Sigma n_1l_1+2vw\Sigma m_1n_1$
$=u^2+v^2+w^2$. (Art. 5·4)

4. **Three straight lines mutually at right angles meet in *a* point *P*, and** two of them **intersect the axes of x and y respectively,** while the third **passes** through a fixed point (0, 0, c) on the z-axis. Show that the equation of the locus of P is $x^2+y^2+z^2=2cz$. *(Delhi Hons., 1949; Punjab, 1963)*

6

The Sphere

6·1. Sphere : Def. A sphere is the locus of a moving point which moves in space such that its distance from a fixed point, called the **centre** of the sphere, is equal to a fixed distance, called the **radius** of the sphere.

SECTION I

STANDARD FORMS

6·2. Standard equation of the sphere.

To find the equation of the sphere whose centre is at the origin and whose radius is a.

Let O be the centre of the sphere of radius a.

Let $P(x, y, z)$ be **any** point on the sphere. Let us take O to be the origin of reference.

$\therefore \quad \overrightarrow{OP}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}.$

$\therefore \quad x^2+y^2+z^2=\overrightarrow{OP}^2=a^2$, which is the required equation of the sphere.

Aliter.

Let O be the centre of a sphere of radius a.

Let P (λ, μ, ν) be **any** point on the sphere. Join OP.

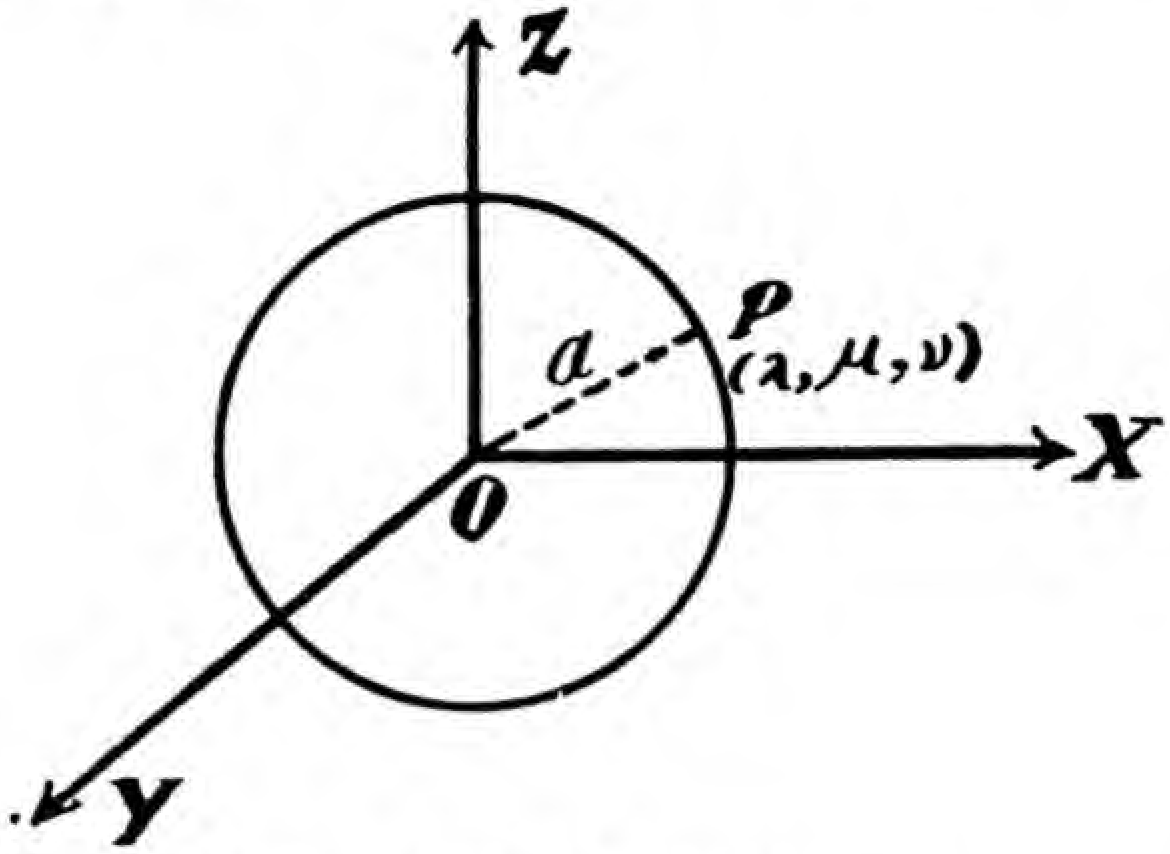
$\therefore \quad OP=a,$

or, $\quad OP^2=a^2,$

or, $\quad (\lambda-0)^2+(\mu-0)^2+(\nu-0)^2=a^2,$

or, $$\lambda^2+\mu^2+\nu^2=a^2.$$

$\therefore$ locus of (λ, μ, ν) is $\mathbf{x}^2+\mathbf{y}^2+\mathbf{z}^2=\mathbf{a}^2$, which is the required equation of the sphere.



Note. Centre and radius of the sphere.

(i) The coordinates of the centre are (0, 0, 0) and (ii) the radius is a.

6·3. Central form of the equation of the sphere.

To find the equation of the sphere whose centre is at the point (a, b, c) and whose radius is r.

Let O be the origin of reference.

Let $c(a, b, c)$ be the centre of the sphere of radius r

$$\therefore \quad \overrightarrow{OC}=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}.$$

Let P be **any** point (x, y, z) on the sphere.

$$\therefore \quad \overrightarrow{OP}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$$

$$\therefore \quad \overrightarrow{CP}=(x-a)\mathbf{i}+(y-b)\mathbf{j}+(z-c)\mathbf{k}$$

$\therefore$ $(x-a)^2+(y-b)^2+(z-c)^2=\text{CP}^2=r^2$, which is the required equation of the sphere.

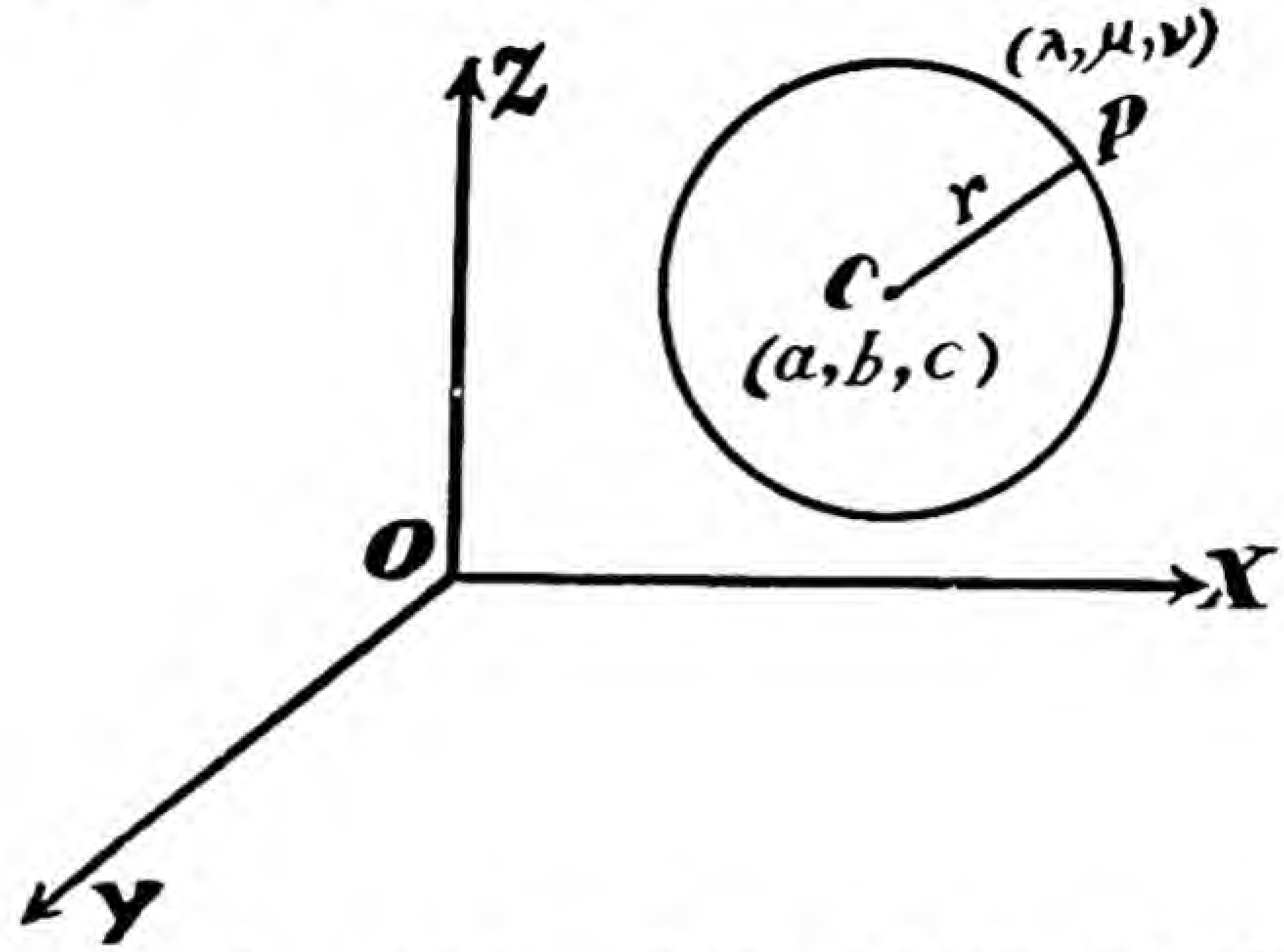
Note. If $\mathbf{r}$ be the position vector of any point on the sphere whose centre has the position vector $\mathbf{c}$ and radius a, then the equation of the sphere is $(\mathbf{r}-\mathbf{c})^2=\mathbf{a}^2$, $\mathbf{r}^2-2\mathbf{r}.\mathbf{c}+k=0$, where $k=c^2-a^2$.

Aliter.

Let $C(a, b, c)$ be the centre and r be the radius of the sphere.

Let $P(\lambda, \mu, \nu)$ be **any** point on the sphere. Join CP.

$\therefore$ $CP=r$, or, $CP^2=r^2$, or, $(\lambda-a)^2+(\mu-b)^2+(\nu-c)^2=r^2$.



$\therefore$ locus of (λ, μ, ν) is $(\mathbf{x}-\mathbf{a})^2+(\mathbf{y}-\mathbf{b})^2+(\mathbf{z}-\mathbf{c})^2=\mathbf{r}^2$, which is the required equation of the sphere.

6·4. General form of the equation of the sphere.

To prove that the equation

$$\mathbf{x^2+y^2+z^2+2ux+2vy+2wz+d=0}$$

represents a sphere for all values of u, v, w and d.

Proof. The given equation is

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0.$$

or, $(x^2+2ux)+(y^2+2vy)+(z^2+2wz)=-d$,

or, $[x-(-u)]^2+[y-(-v)]^2+[z-(-w)]^2=(\sqrt{(u^2+v^2+w^2-d)})^2$

(Note this step.)

This represents the locus of a point (x, y, z) which moves so that its distance from the fixed point $(-u, -v, -w)$ is constant, and equal to $\sqrt{u^2+v^2+w^2-d}$.

$\therefore$ by def. Art. 6·1, it represents a sphere.

This proves the proposition.

Note. 1. Centre and radius.

(i) The centre is $(-u, -v, -w)$,

(ii) the radius is $\sqrt{u^2+v^2+w^2-d}$.

Note 2. Conditions for a sphere.

The general equation of the sphere is

$$x^2+y^2+2ux+2vy+2wz+d=0.$$

$$\therefore \quad ax^2+ay^2+2aux+2avy+2awz+ad=0$$

also represents a sphere.

$\therefore$ *conditions for an equation to represent a sphere are* :

(*i*) *It must be a second degree equation in x, y, and z.*

(*ii*) *The coefficients of* x^2, y^2 *and* z^2 *must be equal.*

(*iii*) *There should be no terms of xy, yz, and zx.*

Note 3. Nature of the sphere.

The sphere is a real sphere, or point sphere, or imaginary sphere according as

$$u^2+v^2+w^2-d \gtreqless 0\ ;$$

when $u^2+v^2+w^2-d<0$, *the sphere is called a* **virtual sphere.**

6·5. Diameter form of the equation of the sphere. To find the equation of the sphere drawn on the join of (x_1, y_1, z_1) and (x_2, y_2, z_2) as diameter.

Let the position vectors of A (x_1, y_1, z_1) and B (x_2, y_2, z_2) be $\mathbf{a}$ and $\mathbf{b}$ respectively. Let $\mathbf{r}$ be the position vector of any point $P(x, y, z)$ on the sphere drawn on AB as diameter.

$$\mathbf{a}=x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k},\ \mathbf{b}=x_2\mathbf{i}+y_2\mathbf{j}+z_2\mathbf{k},\ \mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$$

$$\because \quad AP \perp BP, \quad \therefore \quad \vec{AP}\,.\,\vec{BP}=0,$$

or, $$(\mathbf{r}-\mathbf{a})\,.\,(\mathbf{r}-\mathbf{b})=0,$$

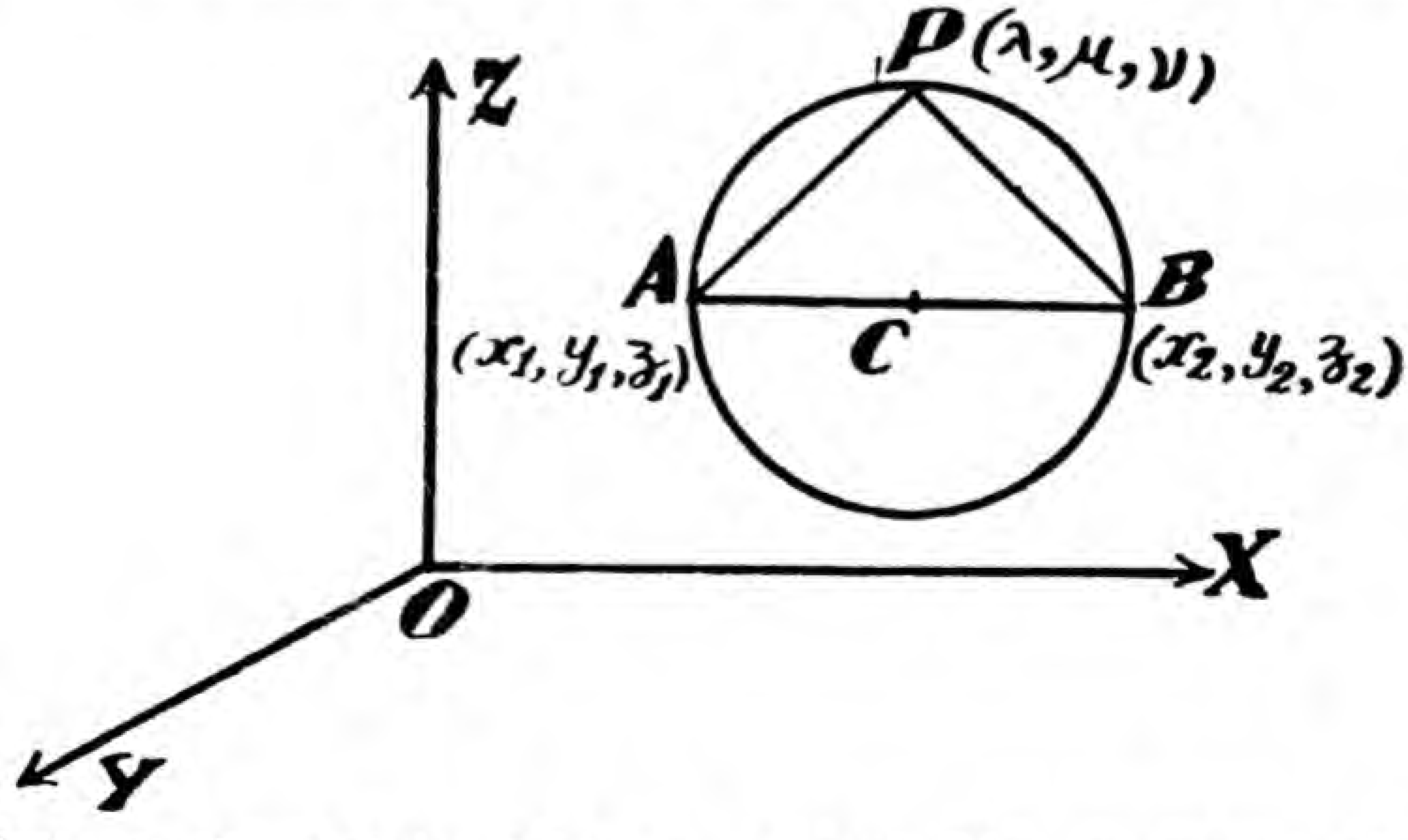
or, $$\{(x-x_1)\,\mathbf{i}+(y-y_1)\mathbf{j}+(z-z_1)\,\mathbf{k}\}\,.\,\{(x-x_2)\,\mathbf{i}+(y-y_2)\,\mathbf{j}+(z-z_2)\,\mathbf{k}\}=0,$$

or, $$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0,$$

which is the required equation.

Aliter.

Let $P(\lambda, \mu, \nu)$ be **any** point on the sphere described on the line



AB joining the points A (x_1, y_1, z_1) and $B(x_2, y_2, z_2)$ as diameter. Join PA and PB.

Now $$\angle APB = 90°,$$

i.e., $$AP \perp BP \qquad \ldots(1)$$

Direction cosines of the line AP are proportional to

$$\lambda - x_1,\ \mu - y_1 \text{ and } \nu - z_1.$$

Also, the direction cosines of the line BP are proportional to

$$\lambda - x_2,\ \mu - y_2 \text{ and } \nu - z_2$$

$\therefore$ from (1), we have

$$(\lambda - x_1)(\lambda - x_2) + (\mu - y_1)(\mu - y_2) + (\nu - z_1)(\nu - z_2) = 0,$$

$\therefore$ locus of (λ, μ, ν) is

$$(\mathbf{x} - \mathbf{x}_1)(\mathbf{x} - \mathbf{x}_2) + (\mathbf{y} - \mathbf{y}_1)(\mathbf{y} - \mathbf{y}_2) + (\mathbf{z} - \mathbf{z}_1)(\mathbf{z} - \mathbf{z}_2) = \mathbf{o},$$

which is the required equation of the sphere.

6·6. Four point form of the equation of the sphere. To find the equation of the sphere passing through four given points, (x_1, y_1, z_1) ; (x_2, y_2, z_2) ; (x_3, y_3, z_3) and (x_4, y_4, z_4).

Let the required sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \qquad \ldots(1)$$

$\because$ (1) passes through (x_1, y_1, z_1),

$\therefore$ $x^2{}_1 + y^2{}_1 + z^2{}_1 + 2ux_1 + 2vy_1 + 2wz_1 + d = 0 \qquad \ldots(2)$

$\because$ (1) passes through (x_2, y_2, z_2),

$\therefore$ $x^2{}_2 + y^2{}_2 + z^2{}_2 + 2ux_2 + 2vy_2 + 2wz_2 + d = 0 \qquad \ldots(3)$

$\because$ (1) passes through (x_3, y_3, z_3),

$\therefore$ $x^2{}_3 + y^2{}_3 + z^2{}_3 + 2ux_3 + 2vy_3 + 2wz_3 + d = 0 \qquad \ldots(4)$

$\because$ (1) passes through (x_4, y_4, z_3),

$\therefore$ $x^2{}_4 + y^2{}_4 + z^2{}_4 + 2ux_4 + 2vy_4 + 2wz_4 + d = 0 \qquad \ldots(5)$

Eliminating u, v, w and d from (1), (2), (3), (4) and (5), we have

$$\begin{vmatrix} x^2 + y^2 + z^2 & x & y & z & 1 \\ x^2{}_1 + y^2{}_1 + z^2{}_1 & x_1 & y_1 & z_1 & 1 \\ x^2{}_2 + y^2{}_2 + z^2{}_2 & x_2 & y_2 & z_2 & 1 \\ x^2{}_3 + y^2{}_3 + z^2{}_3 & x_3 & y_3 & z_3 & 1 \\ x^2{}_4 + y^2{}_4 + z^2{}_4 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0,$$

which is the required equation of the sphere.

EXAMPLES VI (A)

Type I. Ex. 1. Find the equation of the sphere whose centre is (2, −3, 4) ane radius 5. *(Punjab Hons., 1952)*

Sol. The required sphere is $(x-2)^2+(y+3)^2+(z-4)^2=25$,

or, $$x^2+y^2+z^2-4x+6y-8z+4=0.$$

Ex. 2. Find the equation of the sphere whose

(*i*) Centre is $(-6, 1, 3)$ and radius 4

(*ii*) Centre is $(\frac{1}{2}, -\frac{1}{2}, 1)$ and radius 2.

(A.M.I.E., May, 1957 ; Raj. Engg., 1956)

[**Ans.** $x^2+y^2+z^2+12x-2y-6z+30=0$;
$2x^2+2y^2+2z^2-2x+2y-4z-5=0$.]

Type II. Ex. 1. Find the centre and radius of the sphere given by $x^2+y^2+z^2-2x+4y-6z=11$. *(A.M.I.E., May 1955)*

Sol. The sphere is $x^2+y^2+z^2-2x+4y-6z-11=0$,

or, $$(x^2-2x)+(y^2+4y)+(z^2-6z)-11=0.$$

or, $$(x-1)^2-1+(y+2)^2-4+(z-3)^2-9-11=0,$$

or, $$(x-1)^2+[y-(-2)]^2+(z-3)^2=(5)^2.$$

$\therefore$ its centre is $(1, -2, 3)$ and radius is 5.

Short-cut. Here $u=-1$, $v=2$, $w=-3$, $d=-11$.

$\therefore$ centre is $(-u, -v, -w)$, or $(1, -2, 3)$ and radius is

$$\sqrt{u^2+v^2+w^2-d}, \quad \text{or,} \quad \sqrt{1+4+9+11}, \quad \text{or,} \quad 5.$$

Caution. While applying the short-cut method, care must be taken to make the coefficients of x^2, y^2 and z^2 each unity.

Ex. 2. Find the centre and radius of the following spheres

(*i*) $x^2+y^2+z^2-4x+6y-8z+4=0$.

(*ii*) $x^2+y^2+z^2-4x+2y-2z-10=0$.

[**Ans.** $(2, -3, 4)$, 5 ; $(2, -1, 1)$, 4.]

Type III. [Equations of the sphere through four points.]

Ex. 1. Find the equation of the sphere passing through the points (0, 0, 0), (a, 0, 0), (0, b, 0) and (0, 0, c). *(Punjab B.Sc., 1960 ; Calcutta, 1961)*

Sol. Let the required sphere be $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ ...(1)

$\because$ (1) passes through $(0, 0, 0)$,

$\therefore$ $$d=0 \qquad ...(2)$$

$\because$ (1) passes through $(a, 0, 0)$,

$\therefore$ $$a^2+2ua=0, \text{ using (2)},$$

or $$u=-\frac{a}{2}.$$

$\because$ (1) passes through $(0, b, 0)$,

$\therefore$ $$b^2+2vb=0, \text{ using (2)},$$

or, $$v=-\frac{b}{2}.$$

$\because$ (1) passes through $(0, 0, c)$,

$\therefore$ $\quad c^2+2wc=0$, using (2),

or $\quad w=-\dfrac{c}{2}$.

Substituting these values of u, v, w and d in (1), we have

$$\mathbf{x^2+y^2+z^2-ax-by-cz=0},$$

which is the required equation.

Note. Important.

In all subsequent problems, whenever a sphere passing through points (0, 0, 0), (a, 0, 0), (0, b, 0) and (0, 0, c) occurs, we shall directly write its equation as $x^2+y^2+z^2-ax-by-cz=0$.

Ex. 2. Find the equation of the sphere through the points :

(*i*) $(0, 0, 0), (0, 1, -1), (-1, 2, 0)$ and $(1, 2, 3)$. *(Punjab, 1963 ; Calcutta, 1963 ; A.M.I.E., May 1960 : Bombay, 1951)*

(*ii*) $(3, 0, 2), (-1, 1, 1)$ $(2, -5, 4), (-3, 0, 2)$

(*iii*) $(2, 3, 6), (2, 3, 0), (2, 0, 6), (0, 3, 6)$.

[**Ans.** (*i*) $7(x^2+y^2+z^2)-15x-25y-11z=0$;
(*ii*) $x^2+y^2+z^2+4y-6z-1=0$;
(*iii*) $x^2+y^2+z^2-2x-3y-6z=0$]

Ex. 3. Prove that the equation to the sphere circumscribing the tetrahedron whose sides are

$$\frac{y}{b}+\frac{z}{c}=0,\quad \frac{z}{c}+\frac{x}{a}=0,$$

$$\frac{x}{a}+\frac{y}{b}=0,\quad \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1,$$

is

$$\frac{x^2+y^2+z^2}{a^2+b^2+c^2}-\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)=0.$$

[Punjab (Pakistan), 1925S]

[**Hint.** *Solve the equations of the three sides at a time to obtain the four vertices. Proceed as above.*]

Ex. 4. OA, OB, OC are mutually perpendicular lines through the origin, and their direction cosines are

$$l_1, m_1, n_1\ ;\ l_2, m_2, n_2 \text{ and } l_3, m_3, n_3.$$

If $OA=a$, $OB=b$, $OC=c$, prove that the equation of the sphere OABC is
$x^2+y^2+z^2-x(al_1+bl_2+cl_3)-y(am_1+bm_2+cm_3)-z(an_1+bn_2+cn_3)=0$.
(Karnatak, 1961, Punjab, 1952)

Ex. 5. Find the centre of the circle through the points $(-1, 0, 0)$, $(0, 2, 0)$, $(0, 0, 3)$. *(Karnatak Engg., 1956)*

$$\left[\textbf{Ans.}\quad \left(-\frac{13}{98}, \frac{40}{49}, \frac{135}{98}\right).\right]$$

Ex. 6. Find the equation of the sphere which passes through the points $(1, -3, 4)$, $(1, -5, 2)$, and $(1, -3, 0)$ and whose centre lies on the plane $x+y+z=0$. *(Lucknow (Pass), 1960)*

[**Ans.** $x^2+y^2+z^2-2x+6y-4z+10=0$.]

(*ii*) Find the equation of the sphere passing through the points

$$(3, 0, 2), (-1, 1, 1), (2, -5, 4)$$

and having its centre on the plane

$$2x+3y+4z=6.$$ *(Agra Engg., 1962)*

[**Ans.** $x^2+y^2+z^2+4y-6z-1=0$.]

Type IV. Ex. 1. A plane passes through a fixed point (a, b, c) and cuts the axes in A, B, C. Show that the locus of the centre of the sphere OABC. is

$$\frac{\mathbf{a}}{\mathbf{x}}+\frac{\mathbf{b}}{\mathbf{y}}+\frac{\mathbf{c}}{\mathbf{z}}=\mathbf{2}.$$

(Delhi Hons., 1960 ; Lucknow, 1953 ; Kashmir, 1956 ; Punjab, 1961S ; Vikram Engg., 1959 ; Agra, 1962 ; Raj., 1960 ; A M.I.E., Nov., 1957 ; Gujarat Engg., 1954 ; Kashmir, 1956 ; Vikram, 1961 ; Burdwan, 1964 ; Gauhati, 1959)

Sol. Let the plane be $\dfrac{x}{\alpha}+\dfrac{y}{\beta}+\dfrac{z}{\gamma}=1.$ ...(1)

$\because$ (1) passes through (a, b, c),

$$\therefore \qquad \frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}=1 \qquad ...(2)$$

The equation of the sphere OABC is

$$x^2+y^2+z^2-\alpha x-\beta y-\gamma z=0 \qquad ...(3)$$

(Type III, Ex. 1, above)

Let the centre of (3) be (λ, μ, ν).

$$\therefore \qquad \lambda=\frac{\alpha}{2},\ \mu=\frac{\beta}{2},\ \text{and}\ \nu=\frac{\gamma}{2} \qquad ...(4)$$

Eliminating α, β, γ from (2) and (4), we have

$$\frac{a}{2\lambda}+\frac{b}{2\mu}+\frac{c}{2\nu}=1.$$

$\therefore$ locus of (λ, μ, ν) is $\dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z}=2.$

Ex. 2. A sphere of constant radius k passes through the origin and meets the axes in A, B, C. Prove that the centroid of the triangle ABC lies on the sphere $9(x^2+y^2+z^2)=4k^2$.

(Jodhpur, 1964 ; Kashmir, 1953 ; Lucknow, 1962 ; Raj., 1961)

[**Hint.** *Let sphere be* $x^2+y^2+z^2-ax-by-cz=0$.

$\therefore \ a^2+b^2+c^2=4k^2$ ...(1)

Let the centroid be (λ, μ, ν).

$\therefore \ \lambda=\frac{1}{3}a,\ \mu=\frac{1}{3}b$ *and* $\nu=\frac{1}{3}c$ ...(2)

Eliminate a, b, c *between* (1) *and* (2).]

Ex. 3. A sphere of constant radius r passes through the origin O, and cuts the axes in A, B, C. Prove that the locus of the foot of the perpendicular from O to the plane ABC is given by $(x^2+y^2+z^2)^2(x^{-2}+y^{-2}+z^{-2})=4r^2$.

(*Banaras, 1955 ; Sagar B.Sc, 1962*)

Ex. 4. Find the locus of the centre of the variable sphere which passes through the origin O and meets the axes in A, B, C, so that the volume of the tetrahedron OABC is constant.

[**Ans.** xyz=constant.]

Type V. Ex. 1. Find the equation of the sphere on join of (2, —3, 1) and (1, —2, —1) as diameter.

Sol. The required equation of the sphere is

$$(x-2)(x-1)+(y+3)(y+2)+(z-1)(z+1)=0,$$

or,
$$x^2+y^2+z^2-3x+5y+7=0.$$

Ex. 2. Find the equation of the sphere on the join of the following points as diameter :

(*i*) (2, —1, 4), (—2, 2, —2).

(*A.M.I.E, May, 1962 ; Bombay, 1955 ; Sind, 1949*)

(*ii*) (3, —4, 5) and (2, 7, 6). (*Bombay, 1950*)

(*iii*) (1, —1, —1), (—3, 4, 5).

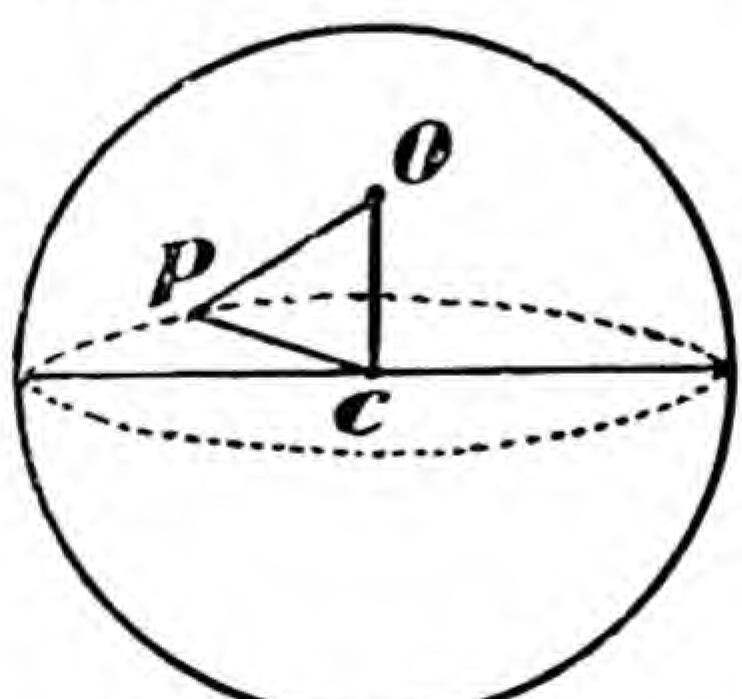
[**Ans.** (*i*) $x^2+y^2+z^2-y-2z-14=0$;
(*ii*) $x^2+y^2+z^2-5x-3y-11z+8=0$;
(*iii*) $x^2+y^2+z^2+2x-3y-4z-12=0$.

SECTION II

THE CIRCLE

6·6. Plane Section of a sphere.

To show that the section of a sphere by a plane is a circle and to find its radius and centre.



Let O be the centre of the sphere and P, any point on the dotted plane section.

Draw OC perpendicular to the given plane, C being the foot of the perpendicular.

$\because$ PC is any line lying in the plane and OC is perpendicular to the plane,

$\therefore$ OC is perpendicular to every line lying in the plane and as such $OC \perp CP$,

$\therefore$ $OP^2 = OC^2 + PC^2$, or, $PC^2 = OP^2 - OC^2$,

or $PC = \sqrt{OP^2 - OC^2}$ = constant, because OP is equal to the radius of the sphere and OC is the perpendicular from the centre of the sphere on the given plane.

$\therefore$ for all positions of P, CP is constant

$\therefore$ the locus of this point P is a circle whose centre is at the point C and whose radius is CP.

Note 1. Centre and radius of the circle.

(**i**) *The* **centre** *of the circle is the foot of the perpendicular from the centre of the sphere on the given plane.*

(**ii**) *The* **radius** *of the circle is*

$\sqrt{(\text{radius of sphere})^2 - (\perp \text{ from the centre of the sphere on the given plane})^2}$.

Note 2. *Such a circle is called a* **small circle.**

Note 3. Great circle : Def.

The section of a sphere by a plane through the centre of the sphere is called a **great circle.**

Its **centre and radius** *are the* **same** *as those of the sphere.*

Note 4. Important,

Two equations, one of a sphere and the other of a plane tagether constitutes the equation of a circle.

$\therefore$ *the circle is given by the equations*

$$x^2+y^2+z^2+2ux+2vy+2wz+d = 0$$

and

$$ax+by+cz+d_1 = 0$$

6·7. Intersection of two spheres.

To show that the curve of intersection of two spheres is a circle.

Proof. Let the two spheres be

$$S_1 \equiv x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1 = 0 \qquad ...(1)$$

and

$$S_2 \equiv x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2 = 0 \qquad ...(2)$$

The coordinates of points on the curve of intersection of (1) and (2) satisfy both the equations, and also satisfy the equation

$$S_1 - S_2 = 0, \text{ or, } 2(u_1-u_2)x+2(v_1-v_2)y+2(w_1-w_2)z + (d_1-d_2) = 0 \quad ...(3)$$

This equation (3) is the equation of the first degree in x, y and z, and therefore represents a plane.

$\therefore$ the curve of intersection of (1) and (2) is the same as that of (1) and (3) or (2) and (3).

$\therefore$ by Art. 6·6, the curve of intersection of two spheres is a circle. This proves the proposition.

6·8. To find the equation of any sphere through the given circle $\left.\begin{array}{r}\mathbf{x^2+y^2+z^2+2ux+2vy+2wz+d=0}\\ \mathbf{lx+my+nz+p=0}\end{array}\right\}$.

The equations of the sphere and the plane are

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad ...(1)$$

and $$lx+my+nz+p=0 \quad ...(2)$$

Let us consider the equation

$$(x^2+y^2+z^2+2ux+2vy+2wz+d)+\lambda(lx+my+nz+p)=0, \quad ...(3)$$

where λ is any constant.

Now, (3) is a second degree equation in x, y, z in which the coefficients of x^2, y^2, z^2 are all equal and there are no terms of yz, zx and xy.

$\therefore$ (3) represents a sphere.

Moreover, equation (3) is satisfied by the coordinates of the points which satisfy both (1) and (2),

i.e., the points common to (1) and (2) lie on the sphere (3).

Hence (3) represents the equation of any sphere which passes through the circle in which the sphere (1) is cut by the plane (2).

Note 1. *λ is determined from the additional condition given in the problem.*

Note 2. *$S_1+\lambda S_2=0$ also reresents a sphere through the circle given by the spheres $S_1=0$, $S_2=0$.*

EXAMPLES VI (B)

Type I. Ex. 1. In Ex. 2 (i), Type I of Examples VI (A), find the area of the section in which the sphere is cut by the plane $\mathbf{x-y+2z+5=0}$.

(A.M.I.E., May, 1957 ; Raj, Engg. 1956)

Sol. The equation of the sphere having centre $(-6, 1, 2)$ and radius 4 is

$$(x+6)^2+(y-1)^2+(z-3)^2=16,$$

or,
$$x^2+y^2+z^2+12x-2y-6z+30=0. \quad \ldots(1)$$

Let r be the radius nf the circle in which the sphere (1) is cut by the plane $x-y+2z+5=0$,

$\therefore\ r=\sqrt{(\text{radius of the sphere})^2-(\perp \text{ from the centre of the sphere on the given plane})^2}$

$$=\sqrt{(4)^2-\left(\frac{-6-1+6+5}{\sqrt{1^2+1^2+2^2}}\right)^2}=\sqrt{16-\frac{16}{6}}=4\sqrt{\frac{5}{6}}.$$

$\therefore$ Area of the circle $=\pi r^2=\pi\,.\,16.\,\dfrac{5}{6}=\dfrac{40\pi}{3}$ sq. units.

Ex. 2. In Ex. 2 (*i*), Type V, Examples VI (A), find the area of the circle in which the sphere is intersected by the plane $2x+y-z=3$.

(*A.M.I.E., May, 1962 ; Sind, 1949*)

$\left[\text{ Ans. } \dfrac{317\pi}{24}\right]$

Type II. Ex. 1. The plane ABC, whose equation is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1,$$

meets the axes in A, B, C. Find the equations to determine the circumcircle of the $\triangle$ ABC, and obtain the coordinates of its centre.

(*Nagpur T.D.C., 1962 ; Lucknow, 1954 ; Pakistan, 1956 ; Allahabad, 1960 ; Kashmir, 1956 ; Punjab, 1957 S ; Gauhati, 1952*)

Sol. The required circumcircle of the $\triangle$ ABC is the circle of intersection of the sphere OABC and the plane ABC.

The plane
$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \quad \ldots(1)$$

cuts the axes in the points A $(a, 0, 0)$, B $(0, b, 0)$ and C $(0, 0, c)$.

Let the sphere OABC be $x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad \ldots(2)$

$\because$ (2) passes through $(0, 0, 0)$ $\quad\therefore\ d=0 \quad \ldots(3)$

$\because$ (2) passes through $(a, 0, 0)$, $\quad\therefore\ a^2+2au=0$, using (3), or, $u=-a/2$.

$\because$ (2) passes through $(0, b, 0)$, $\quad\therefore\ b^2+2bv=0$, using (3), or, $v=-b/2$.

$\because$ (2) passes through $(0, 0, c)$, $\quad\therefore\ c^2+2wc=0$, using (3), or, $w=-c/2$.

$\therefore$ (2) becomes $x^2+y^2+z^2-ax-by-cz=0 \quad \ldots(4)$

Hence the required circumcircle of the $\triangle$ ABC is

$$x^2+y^2+z^2-ax-by-cz=0,$$
$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$$

[**To find the centre of the circumcircle.**]

The centre of tho circle is the foot of the perpendicular from the centre of the sphere on the given plane.

The direction cosines of the perpendicular from the centre of the sphere on the plane (1) are proportional to

$$\frac{1}{a},\ \frac{1}{b},\ \frac{1}{c}.$$

$\therefore$ equations of the perpendicular from the centre of the sphere $\left(\frac{a}{2},\ \frac{b}{2},\ \frac{c}{2}\right)$ on the plane (1) are

$$\frac{x-a/2}{1/a}=\frac{y-b/2}{1/b}=\frac{z-c/2}{1/c}=p,\ \text{say}.$$

$\therefore$ any point on this perpendicular is

$$\left(\frac{a}{2}+\frac{p}{a},\ \frac{b}{2}+\frac{p}{b},\ \frac{c}{2}+\frac{p}{c}\right),$$

or

$$\left(\frac{a}{2}+pa^{-1},\ \frac{b}{2}+pb^{-1},\ \frac{c}{2}+pc^{-1}\right) \qquad \ldots(5)$$

If it lies on the plane (1),

then $$\frac{1}{a}\left(\frac{a}{2}+pa^{-1}\right)+\frac{1}{b}\left(\frac{b}{2}+pb^{-1}\right)+\frac{1}{c}\left(\frac{c}{2}+pc^{-1}\right)=1,$$

or, $$p(a^{-2}+b^{-2}+c^{-2})=-\tfrac{1}{2},$$

or, $$p=-\frac{1}{2(a^{-2}+b^{-2}+c^{-2})}.$$

Substituting this value of p in (5), we get the centre of the required circle as

$$\left[\frac{a}{2}-\frac{a^{-1}}{2(a^{-2}+b^{-2}+c^{-2})},\ \frac{b}{2}-\frac{b^{-2}}{2(a^{-2}+b^{-2}+c^{-2})},\ \frac{c}{2}-\frac{c^{-1}}{2(a^{-2}+b^{-2}+c^{-2})}\right],$$

or, $$\left[\frac{a(b^{-2}+c^{-2})}{2(a^{-2}+b^{-2}+c^{-2})},\ \frac{b(c^{-2}+a^{-2})}{2(a^{-2}+b^{-2}+c^{-2})},\ \frac{c(a^{-2}+b^{-2})}{2(a^{-2}+b^{-2}+c^{-2})}\right].$$

Ex. 2. Find the centre and radius of the circle

$$x^2+y^2+z^2+12x-12y-16z+111=0,$$
$$2x+2y+z=17.$$

(Gauhati, 1958, first part)

[**Ans.** $(-4, 8, 9)$; 4.]

Ex. 3. Find the centre and radius of the circle in which the sphere $x^2+y^2+z^2+2x-2y-4z-19=0$ is cut by the plane

$$x+2y+2z+7=0.$$

(Jodhpur Engi., 1965)

[**Ans.** $(-7/3, -10/3, -4/3)$; 3]

Type III. Ex. 1. If r is the radius of the circle

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0, \quad lx+my+nz=9,$$

prove that $$(r^2+d^2)(l^2+m^2+n^2)=(mw-nv)^2+(nu-lw)^2+(lv-mu)^2.$$

[*Raj., (Physics), 1963 ; Allahabad, 1962*]

Sol. The centre of the sphere is $(-u, -v, -w)$ and the radius is

$$\sqrt{u^2+v^2+w^2-d}.$$

Perpendicular from $(-u, -v, -w)$ on the plane $lx+my+nz=0$ is

$$\frac{lu+mv+nw}{\sqrt{l^2+m^2+n^2}}.$$

$$\therefore \quad r^2=(u^2+v^2+w^2-d)-\frac{(lu+mv+nw)^2}{l^2+m^2+n^2},$$

or $$(r^2+d)(l^2+m^2+n^2)=(l^2+m^2+n^2)(u^2+v^2+w^2)-(lu+mv+nw)^2$$
$$=(mw-nv)^2+(nu-lw)^2+(lv-mu)^2$$

, using Lagrange's identity.

Ex. 2. Show that the radius of the circle
$x^2+y^2+z^2+x+y+z-4=0, \quad x+y+z=0$ is 2.

(*Kashmir, 1957 ; Punjab, 1958 S*)

Ex. 3. Find the centre and radius of the circle in which the sphere $x^2+y^2+z^2=9$ is cut by the plane $x+y+z=1$. (*Raj., Engg., 1960*)

$$\left[\textbf{Ans. } \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), \sqrt{\frac{26}{3}}\right)\right]$$

Type IV. Ex. 1. A is a point on OX and B on OY so that the angle OAB is constant $(=\alpha)$. On AB as diameter a circle is described whose plane is parallel to OZ. Prove that as AB varies the circle generates the cone $2xy-z^2 \sin 2\alpha=0$. (*Agra, 1963*)

Sol. Let OA be $2a$ and OB be $2b$.

$\therefore$ Coordinates of A and B are respectively $(2a, 0, 0)$ and $(0, 2b, 0)$.

$\because$ $\angle OAB=\alpha$,

$$\therefore \qquad \tan\alpha=\frac{b}{a} \qquad \ldots(1)$$

Now, the sphere on AB as diameter is

$$(x-2a)(x-0)+(y-0)(y-2b)+(z-0)(z-0)=0,$$

or, $$x^2+y^2+z^2-2ax-2by=0 \qquad \ldots(2)$$

The plane through AB parallel to OZ is

$$\frac{x}{2a}+\frac{y}{2b}=1^* \qquad \ldots(3)$$

The circle in question is the curve of intersection of (2) and (3).

To find the locus of this circle, we shall eliminate a and b between (2) and (3) with the help of (1).†

From (3) and (1), we have $2a \tan\alpha = x\tan\alpha + y$, .. (4), on eliminating b.

From (1) and (2) we have, on eliminating b,

$$(x^2+y^2+z^2)-2(ax+ay\tan\alpha)=0 \qquad \ldots(5)$$

Eliminating a between (4) and (5), we have

$$\tan\alpha(x^2+y^2+z^2)-(x+y\tan\alpha)(x\tan\alpha+y)=0,$$

or, $\quad (x^2+y^2+z^2)\tan\alpha-[xy\sec^2\alpha+(x^2+y^2)\tan\alpha]=0,$

or, $\quad xy\sec^2\alpha-z^2\tan\alpha=0,$ or, $xy\,.\,\dfrac{1}{\cos^2\alpha}-z^2\dfrac{\sin\alpha}{\cos\alpha}=0,$

or, $\quad xy-z^2\sin\alpha\cos\alpha=0,\qquad$ or, $\qquad 2xy-z^2\sin 2\alpha=0.$

Ex 2. POP′ is a variable diameter of the ellipse

$$z=0,\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

and a circle is described in the plane PP′ZZ′ on PP′ as diameter. Prove that as PP′ varies, the circle generates the surface

$$(x^2+y^2+z^2)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)x=x^2+y^2.$$

Type V. Ex. 1. Find the equation to the sphere though the circle

$$\mathbf{x^2+y^2+z^2=9,\ 2x+3y+4z=5}$$

and the point (1, 2, 3).

(A M.I.E. May, 1959 ; Punjab, 1961 ; Pakistan 1954 ; Poona, 1950 ; Karnatak B.Sc., 1962)

---

*Let the plane be $Ax+By+Cz+D=0$. ...(A)

$\because$ it passes through A and B,

$\therefore$ $A.2a+D=0$ and $B.2b+D=0$.

Also, $\because$ plane (A) is parallel to OZ,

$\therefore$ $C=0$ $\quad\therefore$ (A) gives $-\dfrac{dx}{2a}-\dfrac{dy}{2b}+d=0,$

or, $$\frac{x}{2a}+\frac{y}{2b}=1.$$

† The elimination can also be conducted as under :

From (2) and (3), we have $\quad x^2+y^2+z^2-(2ax+2by)\left(\dfrac{x}{2a}+\dfrac{y}{2b}\right)=0,$

or, $\quad x^2+y^2+z^2-(ax+by)\left(\dfrac{x}{a}+\dfrac{y}{b}\right)=0,$

or, $\quad x^2+y^2+z^2-(x^2+y^2)-xy\left(\dfrac{a}{b}+\dfrac{b}{a}\right)=0,$

or, $$z^2-xy(\tan\alpha+\cot\alpha)=0.$$

or, $$z^2\sin 2\alpha-2xy=0.$$

Sol. The equation of the sphere passing through the circle

$x^2+y^2+z^2=9$ and $2x+3y+4z-5=0$ is

$$(x^2+y^2+z^2-9)+\lambda(2x+3y+4z-5)=0 \qquad ...(1)$$

$\because$ (1) passes through (1, 2, 3), $\therefore$ $\lambda=-\frac{3}{2}$.

$\therefore$ (1) becomes $3(x^2+y^2+z^2)-2x-3y-4z-22=0$.

Ex. 2. Find the equation of the sphere which passes through the circle

(*i*) $z=0$, $x^2+y^2=a^2$ and the point (α, β, γ).

(*ii*) $x^2+y^2+z^2-2x=9$, $z=0$ and the point (4, 5, 6). (*Poona, 1951*)

(*A.M.I.E. May, 1958 ; Karnatak, 1950 ; Kashmir, 1951*)

(*iii*) $y^2+z^2=25$, $x=0$ and the point (1, 3, -2). (*Raj. Engg., 1958*)

[**Ans.** (*i*) $\gamma(x^2+y^2+z^2-a^2)=z(\alpha^2+\beta^2+\gamma^2-a^2)$,

(*ii*) $x^2+y^2+z^2-2x-10z-9=0$;

(*iii*) $x^2+y^2+z^2+11x-25=0$.]

Ex. 3. Find the equation of the sphere which passes through the circle $x^2+y^2=4$, $z=0$ and is cut by the plane $x+2y+2z=0$ in a circle of radius 3.

(*A.M.I.E., May, 1958 ; Karnatak Engg., 1954*)

[**Ans.** $x^2+y^2+z^2\pm 6z-4=0$.]

Ex. 4. Prove that the plane $x+2y-z=4$ cuts the sphere $x^2+y^2+z^2-x+z-2=0$ in a circle of radius unity ; and find the equation of a sphere which has this circle for one of the great circles. (*Delhi Hons., 1963*)

[**Ans.** $x^2+y^2+z^2-2x-2y+2z+2=0$.]

Ex. 5. Obtain the equation of the sphere having the circle

$$x^2+y^2+z^2+10y-4z=8,\ x+y+z=3$$

as a great circle. (*Karnatak Engg., 1953 ; Delhi Engg., 1963*)

[**Ans.** $x^2+y^2+z^2-4x+6y-8z+4=0$.]

Ex 6. Find the equation of the sphere through the circle

$$x^2+y^2+z^2+2x+3y+6=0,\ x-2y+4z-9=0$$

and the centre of the sphere $x^2+y^2+z^2-2x+4y-6z+5=0$.

(*Jodhpur Engg., 1965 Sup.*)

[**Ans.** $x^2+y^2+z^2+7y-8z+24=0$.]

Ex. 7. Find the equations of the spheres through the circle

$$x^2+y^2+z^2=1,\ 2x+4x+5z=6$$

and touching the plane $z=0$. (*Burdwan, 1964*)

[**Ans.** $x^2+y^2+z^2-2x-4y-5z+5=0$,

$5x^2+5y^2+5z^2-2x-4y-5z+1=0$.]

Type VI. Ex. 1. Prove that the circles $x^2+y^2+z^2-2x+3y+4z-5=0$, $5y+6z+1=0$; $x^2+y^2+z^2-3x-4y+5z-6=0$, $x+2y-7z=0$ lie on the same sphere, and find its equation. (*Agra, 1951 ; I.A.S., 1953 ; Karnatak, 1961 ; Pakistan, 1953 ; Lucknow (Pass), 1961*)

Sol. The given circles are

$$x^2+y^2+z^2-2x+3y+4z-5=0,\ 5y+6z+1=0 \qquad ...(1)$$

and $$x^2+y^2+z^2-3x-4y+5z-6=0,\ x+2y-7z=0 \qquad ...(2)$$

The equation of any sphere through (1) is

$$x^2+y^2+z^2-2x+3y+4z-5+\lambda(5y+6z+1)=0 \qquad \dots(3)$$

The equation of any sphere through (2) is

$$x^2+y^2+z^2-3x-4y+5z-6+\mu(x+2y-7z)=0 \qquad \dots(4)$$

The circles (1) and (2) will lie on the same sphere, if we can find λ, μ such that (3) and (4) represent the same sphere.

Equating coefficients of like terms in (3) and (4), we have

$$-2=-3+\mu \quad \dots(5), \qquad 3+5\lambda=-4+2\mu \quad \dots(6),$$

$$4+6\lambda=5-7\mu \quad \dots(7), \text{ and } \quad -5+\lambda=-6 \quad \dots(8)$$

From (5) and (8), $\mu=1$, $\lambda=-1$.

These values of λ and μ satisfy (6) and (7).

$\therefore$ (3) and (4) represent the same sphere, when $\lambda=-1$ and $\mu=1$.

Putting $\lambda=-1$ in (3) or $\mu=1$ in (4), we have

$$x^2+y^2+z^2-2x-2y-2z-6=0.$$

Ex. 2. Show that the two circles

$$x^2+y^2+z^2-y+2z=0,$$
$$x-y+z-2=0;$$
$$x^2+y^2+z^2+x-3y+z-5=0,$$
$$2x-y+4z-1=0$$

lie on the same sphere, and find its equation.

(*Delhi Hons., 1947 ; Kashmir, 1955 ; Lucknow (Pass), 1963*)

[**Ans.** $x^2+y^2+z^2+3x-4y+5z-6=0.$]

Ex. 3. Find the conditions that the circles

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0, \qquad lx+my+nz=p,$$
$$x^2+y^2+z^2+2u'x+2v'y+2w'z+d'=0, \qquad l'x+m'y+n'=p'$$

should lie on the same sphere. (*Punjab B.Sc., 1962S*)

[**Ans.**
$$\left\|\begin{matrix} 2(u-u') & 2(v-v') & 2(w-w') & d-d' \\ l & m & n & -p \\ l' & m' & n' & -p' \end{matrix}\right\| = 0,$$

the notation indicating that each of the four determinants obtained by omitting the first, second, third, fourth columns one by one from it is zero.]

SECTION III

A LINE AND A SPHERE. EQUATIONS OF TANGENT PLANES AND CONDITION OF TANGENCY

6·9 To find the points of intersection of the sphere

$$\mathbf{x^2+y^2+z^2=a^2}$$

and the line $$\frac{\mathbf{x}-\alpha}{\mathbf{l}}=\frac{\mathbf{y}-\beta}{\mathbf{m}}=\frac{\mathbf{z}-\gamma}{\mathbf{n}}.$$

Let the sphere and line be

$$x^2+y^2+z^2=a^2 \quad ...(1)$$

and

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r, \text{ say} \quad ...(2)$$

Any point on (2) is

$$(x_1+lr, y_1+mr, z_1+nr) \quad ...(3)$$

If it lies on the sphere (1), then

$$(x_1+lr)^2+(y_1+mr)^2+(z_1+nr)^2=a^2,$$

or

$$r^2(l^2+m^2+n^2)+2r(lx_1+my_1+nz_1)+S_1=0 \quad ...(4)$$

where

$$S_1\equiv x_1^2+y_1^2+z_1^2-a^2.$$

Now, equation (4) is a quadratic equation in r, and therefore gives two values of r, corresponding to each of which (3) gives a point common to (1) and (2). Thus we obtain the coordinates of two points of intersection of (1) and (2).

6·10. Tangent line and tangent plane : Def.

(*ii*) Let P and Q be the two points on a sphere. Let M be the middle point of PQ. Let C be the centre of the sphere. Further, let CM meet the sphere in A. If PQ moves parallel to itself with its middle point M on CA, then when M is at A, PQ is called the **tangent line** to the sphere at A.

(*ii*) The locus of the tangent lines at A is a plane, called the **tangent plane** at A.

6·11. Equation of the tangent plane.

To find the equation of the tangent plane at the point (x_1, y_1, z_1) of the sphere

$$\mathbf{x^2+y^2+z^2+2ux+2vy+2wz+d=0.}$$

The equation of the sphere is

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad ...(1)$$

Equations of any line through (x_1, y_1, z_1) are

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r, \text{ say.} \quad ...(2)$$

Any point on this line is (x_1+lr, y_1+mr, z_1+nr).

If it lies on the sphere (1), then

$$(x_1+lr)^2+(y_1+mr)^2+(z_1+nr)^2+2u(x_1+lr)$$
$$+2v(y_1+mr)+2w(z_1+nr)+d=0,$$

or, $$r^2(l^2+m^2+n^2)+2r[l(x_1+u)+m(y_1+v)+n(z_1+w)]$$
$$+x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d=0 \qquad ...(3)$$

This is a quadratic equation in r.

$\because$ (x_1, y_1, z_1) lies on (1),

$$\therefore \quad x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+z_1+d=0 \qquad ...(4)$$

$\therefore$ one root of the equation (3) is zero.

If the line (2) touches the sphere (1), the other root of (3) is also zero,

i.e., coefficient of $r=0$,

i.e., $$l(x_1+u)+m(y_1+v)+n(z_1+w)=0 \qquad ...(5)$$

The tangent plane or the locus of the tangent lines at (x_1, y_1, z_1) is obtained by eliminating l, m, n between (2) and (5).

$$\therefore \quad (x-x_1)(x_1+u)+(y-y_1)(y_1+v)+(z-z_1)(z_1+w)=0,$$

or, $$xx_1+yy_1+zz_1+ux+vy+wz$$
$$=x_1^2+y_1^2+z_1^2+ux_1+vy_1+wz_1$$

or, $$xx_1+yy_1+zz_1+ux+vy+wz$$
$$=-ux_1-vy_1-wz_1-d, \text{ using (4).}$$

Hence $\mathbf{xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=o}$, which is the required equation of the tangent plane at (x_1, y_1, z_1).

Aid to memory. The equation of the tangent plane at (x_1, y_1, z_1) is obtained by writing xx_1 for x^2, yy_1 for y^2, zz_1 for z^2, $x+x_1$ for $2x$, $y+y_1$ for $2y$ and $z+z_1$ for $2z$ in the equation of the given sphere.

Cor. Tangent plane at a point for a standard sphere.

(*Pakistan, 1958*)

The equation of the tangent plane at the point (x_1, y_1, z_1) of the sphere $$\mathbf{x^2+y^2+z^2=a^2} \text{ is}$$
$$\mathbf{xx_1+yy_1+zz_1=a^2}.$$

Note 1. The tangent plane at any point of a sphere is perpendicular to the radius though that point.

Proof. *The equation of the tangent plane at the point (x_1, y_1, z_1) of the sphere*

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0$$

is $$xx_1+yy_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=0 \qquad ...(1)$$

The direction cosines of the normal to (1) are proportional to x_1+u, y_1+v *and* z_1+w.

The direction cosines of the line joining the centre $(-u, -v, -w)$ *and the point* (x_1, y_1, z_1) *are proportional to* $x_1-(-u)$, $y_1-(-v)$, $z_1-(-w)$, *or to* x_1+u, y_1+v, z_1+w.

$\therefore$ *normal to the plane* (*1*) *is parallel to the radius.*

$\therefore$ *tangent plane at* (x_1, y_1, z_1) *is perpendicular to the radius through* (x_1, y_1, z_1).

Note 2. The tangent line at a point of the sphere is perpendicular to the radius through that point.

Proof. *Let the point be* (x_1, y_1, z_1).

The direction cosines of the tangent line are l, m, n *and the direction cosines of the radius through* (x_1, y_1, z_1) *are proportional to* x_1+u, y_1+v, z_1+w.

$\therefore$ *from equation* (*5*) *above, the tangent line at* (x_1, y_1, z_1) *is perpendicular to the radius through* (x_1, y_1, z_1).

Note 3. Geometrical condition of tangency.

(*i*) *If a plane touches a sphere, the length of the perpendicular from the centre on the plane* $= \pm$ *radius.*

(*ii*) *If a line touches a sphere, the length of the perpendicular from the centre on the line* $= \pm$ *radius.*

6·12. Condition of tangency.

To find the condition that the plane $lx+my+nz=p$ is a tangent plane to the sphere

$$\mathbf{x^2+y^2+z^2+2ux+2vy+2wz+d=0.}$$

(*Delhi Hons., 1955 ; Punjab, B.Sc., 1961 : Kashmir, 1957*)

The centre of the sphere is $(-u, -v, -w)$ and the radius is

$$\sqrt{u^2+v^2+w^2-d}.$$

If the plane

$$lx+my+nz=p \qquad \text{...(1)}$$

touches the given sphere, then the perpendicular from the centre on the plane $= \pm$radius,

or,
$$\frac{lu+mv+nw+p}{\sqrt{l^2+m^2+n^2}} = \pm\sqrt{u^2+v^2+w^2-d},$$

or,
$$(lu+mv+nw+p)^2=(l^2+m^2+n^2)(u^2+v^2+w^2),$$

which is the required condition.

EXAMPLES VI (C)

Type I. Ex. 1. Prove that the sum of the squares of the intercepts ade by a given sphere on any three mutually perpendicular lines through a fixed point is constant.

(Kashmir, 1954 ; Punjab B.Sc., 1962 S)

Sol. Let us choose *the fixed point as the origin and any three mutually perpendicular lines through it as the coordinate axes.*

With reference to these axes, let the equation of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \qquad \ldots(1)$$

The x-axis ($y=0$, $z=0$) meets (1) in points given by

$$x^2+2ux+d=0.$$

Let x_1 and x_2 be its roots.

$\therefore$ $$x_1+x_2=-2u$$
and $$x_1x_2=d.$$

$\therefore$ (intercept on x-axis)^2

$$=(x_1-x_2)^2=(x_1+x_2)^2-4x_1x_2$$
$$=4(u^2-d).$$

Similarly, (intercept on y-axis)$^2=4(v^2-d)$
and (intercept on z-axis)$^2 \quad =4(w^2-d)$.

$\therefore$ sum of the squares of the intercepts

$$=4(u^2+v^2+w^2-3d),$$

which is constant.

Ex. 2. Does the line

$$2x-1=y+3=4-z$$

intersect the sphere

$$x^2+y^2+z^2-6x+8y-4z+4=0\ ?$$

(Bombay, 1952)

$\left[\textbf{Ans.}\right.$ Yes, in points $\left(\frac{7}{3}, \frac{2}{3}, \frac{1}{3}\right)$ and $\left.\left(-\frac{1}{3}, \frac{-14}{3}, \frac{17}{3}\right).\right]$

Ex. 3. Find the points of intersection of the line

$$\frac{x-8}{4}=\frac{y}{1}=-(z-1)$$

and the sphere

$$x^2+y^2+z^2-4x+6y-2z+5=0.$$

[**Ans.** $(4, -1, 2)$, $(0, -2, 3)$.]

Type II. Ex. 1. Find the equations to the spheres which pass through the circle $\mathbf{x^2+y^2+z^2=5,\ x+2y+3z=3}$
and touch the plane $\mathbf{4x+3y=15}$.

(Karnatak, 1961 ; Punjab, B.Sc., 1961 ; Raj. Engi., 1953 ; Kashmir, 1956 ; Punjab, 1957)

Sol. Equation of a sphere passing through the given circle is

$$x^2+y^2+z^2-5+\lambda(x+2y+3z-3)=0 \qquad \ldots(1)$$

Its centre is $\left(-\frac{\lambda}{2}, -\lambda, -\frac{3\lambda}{2}\right)$

and radius is

$$\sqrt{\frac{\lambda^2}{4}+\lambda^2+\frac{9\lambda^2}{4}+5+3\lambda},$$

or,
$$\sqrt{\frac{7\lambda^2}{2}+5+3\lambda}.$$

$\because$ the sphere (1) touches the plane

$$4x+3y-15=0, \qquad \ldots(2)$$

$\therefore$ $\perp$ from $\left(-\frac{\lambda}{2}, -\lambda, -\frac{3\lambda}{2}\right)$ on (2)

$$=\pm\sqrt{\frac{7\lambda^2}{2}+5+3\lambda}$$

or
$$\frac{2\lambda+3\lambda+15}{\sqrt{16+9}}=\pm\sqrt{\frac{7\lambda^2}{2}+5+3\lambda}$$

or
$$25(\lambda+3)^2=25\left(\frac{7\lambda^2}{2}+5+3\lambda\right)$$

or
$$2\lambda^2+12\lambda+18=7\lambda^2+10+6\lambda,$$

or
$$5\lambda^2-6\lambda-8=0,$$

or
$$5\lambda^2-10\lambda+4\lambda-8=0,$$

or
$$5\lambda(\lambda-2)+4(\lambda-2)=0,$$

or
$$(5\lambda+4)(\lambda-2)=0.$$

$\therefore$
$$\lambda=2, -\frac{4}{5}.$$

$\therefore$ (1) gives, when $\lambda=2$,

$$x^2+y^2+z^2+2x+4y+6z-11=0$$

and when $\lambda=-\frac{4}{5}$,

$$5x^2+5y^2+5z^2-4x-8y-12z-13=0.$$

Ex. 2. Find the equations of the spheres which pass through the circle

$$x^2+y^2+z^2-4x-y+3z+12=0,$$
$$2x+3y-7z=10,$$

and touch the plane

$$x-2y+2z=1. \qquad \textit{(Bombay, 1956)}$$

[**Ans.** $x^2+y^2+z^2-2x+2y-4z+2=0$,
$x^2+y^2+z^2-6x-4y+10z+22=0$.]

Ex. 3. Find the equation of the sphere which touches the sphere

$$x^2+y^2+z^2+2x-6y+1=0$$

at the point $(1, 2, -2)$ and passes through the origin.

(Delhi Hons., 1963 ; Punjab, 1950 ; Delhi Engg., 1962)

[**Ans.** $4(x^2+y^2+z^2)+10x-25y-2z=0$.]

Ex. 4. Find the equations of the spheres which pass through the circle

$$x^2+y^2+z^2=1,$$
$$2x+4y+5z=6$$

and touch the plane $z=0$.

(Punjab, 1958 ; Kashmir, 1958)

[**Ans.** $x^2+y^2+z^2-2x-4y-5z+5=0$,
$5(x^2+y^2+z^2)-2x-4y-5z+1=0$.]

Ex. 5. Find the equation of the sphere passing through the circle

$$x^2+y^2-6x-2z+5=0, \quad y=0$$

and touching the plane $3y+4z+5=0$.

[$x^2+y^2+z^2-6x-4y-2z+5=0$; $x^2+y^2+z^2-6x-\frac{11}{4}y-2z+5=0$]

(A.M.I.E., Nov. 1962)

Type III. Ex. 1. Find the equations of the tangent planes to the sphere $x^2+y^2+z^2+2x-4y+6z-7=0$ which intersect in the line $6x-3y-23=0=3z+2$.

(Delhi Hons., 1954 ; Punjab, 1963)

Sol. Any plane through the line $6x-3y-23=0=3z+2$ is

$$6x-3y-23+\lambda(3z+2)=0 \qquad ...(1)$$

It will touch the given sphere

$$x^2+y^2+z^2+2x-4y+6z-7=0,$$

if the perpendicular from the centre $(-1, 2, -3)$ on (1)

$$=\pm\sqrt{1+4+9+7},$$

or if
$$\frac{-6-6-23+\lambda(2-9)}{\sqrt{36+9+9\lambda^2}}=\pm\sqrt{21},$$

or if
$$(-7\lambda-35)^2=(45+9\lambda^2)(21),$$

or if
$$49(\lambda+5)^2=(21)(9)(\lambda^2+5),$$

or if
$$7\lambda^2+70\lambda+175=27\lambda^2+135,$$

or if
$$20\lambda^2-70\lambda-40=0,$$

or if
$$2\lambda^2-7\lambda-4=0, \text{ or if } (2\lambda+1)(\lambda-4)=0$$

$\therefore \quad \lambda=4, -\frac{1}{2}$.

When $\lambda=4$, (1) gives $6x-3y-23+12z+8=0$,

or,
$$6x-3y+12z-15=0,$$

or,
$$2x-y+4z=5 \qquad ...(2)$$

When $\lambda=-\frac{1}{2}$, (1) gives $6x-3y-23-\frac{1}{2}(3z+2)=0$,

or,
$$12x-6y-46-3z-2=0,$$

or,
$$12x-6y-3z-48=0,$$

or,
$$4x-2y-z-16=0 \qquad ...(3)$$

$\therefore$ (2) and (3) are the required tangent planes.

Ex. 2. Find equations of two tangent planes to the sphere $x^2+y^2+z^2=9$, which pass through the line $x+y=6, \quad x-2z=3$.

(Karnatak Engg., 1961)

[**Ans.** $2x+y-2z=9$; $x+2y+2z=9$.]

Ex. 3. Obtain the equations of the tangent planes to the sphere $x^2+y^2+z^2=9$ which can be drawn through the line

$$\frac{x-5}{2}=-\frac{y-1}{2}=\frac{z-1}{1}. \qquad \textit{(Punjab, 1959)}$$

[**Ans.** $x+2y+2z=9$, $2x+y-2z=9$.]

Ex. 4. Find the equation of the tangent plane at the point

$$(a\cos\theta\sin\phi,\quad a\sin\theta\sin\phi,\quad a\cos\phi)$$

to the sphere $\quad x^2+y^2+z^2=a^2$ *(North Bengal, 1963)*

[**Ans.** $x\cos\theta\sin\phi+y\sin\theta\sin\phi+z\cos\phi=a$]

Ex. 5. Find the equations of the two tangent planes of the sphere

$$x^2+y^2+z^2-4x+2y-4=0$$

which are parallel to the plane $2x-y+2z=1$; also find the coordinates of the point of contact. *(Gauhati, 1963)*

[**Ans.** $2x-y+2z+4=0$,
$2x-y+2z=14$.]

Type IV. Ex. 1. (i) Sphere touching the coordinate planes. Find the equation of a sphere touching the three coordinate planes. How many such spheres can be drawn?

(ii) Sphere touching the axes. Find the equation of the sphere touching the three coordinate axes. How many spheres can be so drawn?

Sol. (*i*) Let the required equation of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \qquad ...(1)$$

$\because$ it touches the yz-plane, *i.e.*, $x=0$,
then the condition of tangency gives

$$-\frac{u}{1}=\pm\sqrt{u^2+v^2+w^2-d},$$

or, $$v^2+w^2=d \qquad ...(2)$$

Similarly, $$w^2+u^2=d \qquad ...(3)$$

and $$u^2+v^2=d \qquad ...(4)$$

Adding (2), (3) and (4), we have

$$u^2+v^2+w^2=\frac{3d}{2} \qquad ...(5)$$

Subtracting (2), (3) and (4) in succession from (5), we have

$$u^2=\frac{d}{2},\quad v^2=\frac{d}{2},\quad w^2=\frac{d}{2}.$$

$$\therefore \quad u^2=v^2=w^2=\frac{d}{2}=a^2, \text{ say.}$$

$$\therefore \quad u=\pm a,\ v=\pm a,\ w=\pm a, \text{ and } d=2a^2.$$

$\therefore$ (1) gives $x^2+y^2+z^2\pm 2ax\pm 2ay\pm 2az\pm 2a^2=0$.

There will be an infinite number of spheres depending on the value of a.

However, if a be given, there will be only eight such spheres.

(*ii*) Let the required sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \qquad ...(1)$$

It meets x-axis $(y=0,\ z=0)$, where

$$x^2+2ux+d=0.$$

$\because$ The sphere touches the x-axis,

$\therefore$ $$4u^2=4d,$$

or, $$u^2=d.$$

Similarly, $\because$ The sphere also touches the y and z-axis,

$\therefore$ $$v^2=d, \quad w^2=d.$$

$\therefore$ $$u^2=v^2=w^2=d=a^2, \text{ say.}$$

$\therefore$ $$u=\pm a,\ v=\pm a.\ w=\pm a,\ d=a^2$$

$\therefore$ (1) gives, $x^2+y^2+z^2\pm 2a(x+y+z)+a^2=0.$

There will be an infinite number of such spheres depending on the value of a.

However, if a be given, there will be only 8 such spheres.

Ex. 2. Prove that the equation to a sphere, which lies in the octant $OXYZ$ and touches the coordinate planes, is of the form

$$x^2+y^2+z^2-2\lambda\,(x+y+z)+2\lambda^2=0. \quad \text{(Punjab B.Sc., 1958 S)}$$

Prove that in general two spheres can be drawn through a given point to touch the coordinate planes, and find for what positions of the point the spheres are (*i*) real ; (*ii*) coincident. (*Delhi Hons., 1961*)

Ex. 3. Find the point at which the plane $3y+4z-31=0$ touches the sphere

$$x^2+y^2+z^2-2x+4y-6z=11.$$

(*A.M.I.E., May, 1955*)

[**Ans.** (1, 1, 7).]

Ex. 4. Show that the plane $2x-2y+z+12=0$ touches the sphere

$$x^2+y^2+z^2-2x-4y+2z=3,$$

and find the point of contact. (*Agra Engg., 1962*)

[**Ans.** $(-1, 4, -2)$]

Type V.

Note. (*i*) *Two spheres touch externally, if the distance between their centres is equal to sum of their radii.*

(*ii*) *Two spheres touch internally, if the distance between their centres is equal to the difference of their radii.*

(*iii*) *When the spheres touch externally, the point of contact divides the join of the centres* **internally** *in the ratio of the radii.*

(*iv*) *When the spheres tonch internally, the point of contact divides the join of centres* **externally**, *in the ratio of the radii.*

Ex. 1. Show that the spheres

$$\mathbf{x^2+y^2+z^2=25,}$$
$$\mathbf{x^2+y^2+z^2-18x-24y-40z+225=0}$$

touch and find the coordinates of their common point.

(*Punjab, 1950S, 1962*)

Sol. The spheres are

$$x^2+y^2+z^2=25 \quad ...(1)$$

and $$x^2+y^2+z^2-18x-24y-40z+225=0 \quad ...(2)$$

Centre of (1) is (0, 0, 0) and its radius is 5.

Centre of (2) is (9, 12, 20) and its radius is 20.

$$\begin{aligned} \text{Distance between the centres } &= \sqrt{81+144+400} \\ &= \sqrt{625}=25. \\ &= \text{sum of the radii.} \end{aligned}$$

$\therefore$ the spheres (1) and (2) touch externally. The point of contact divides the join of centres internally in the ratio of their radii.

$\therefore$ its coordinates are

$$\left[\frac{5.9+20.0}{5+20}, \frac{5.12+20.0}{5+20}, \frac{5.20+20.0}{5+20}\right]$$

or

$$\left[\frac{9}{5}, \frac{12}{5}, 4\right].$$

Ex. 2. Show that the spheres

$$x^2+y^2+z^2=64$$

and

$$x^2+y^2+z^2-12x+4y-6z+48=0$$

touch internally, and find the point of contact. *(Punjab, 1953)*

$$\left[\textbf{Ans.}\ \left(\frac{48}{7}, \frac{16}{7}, \frac{24}{7}\right).\right]$$

Ex. 3. Show that the spheres

$$x^2+y^2+z^2=100$$

and

$$x^2+y^2+z^2-24x-30y-32z+400=0$$

touch externally, and find the point of contact.

$$\left[\textbf{Ans.}\ \left(\frac{24}{5}, 6, \frac{32}{5}\right).\right]$$

SECTION IV

PLANE OF CONTACT AND POLAR PLANE

6·13. Plane of contact : Def.

If tangent planes are drawn to a sphere from an external point, then the locus of their points of contact is called the plane of contact.

6·14. To find the equation of the plane of contact of tangent planes drawn through the point (x_1, y_1, z_1) to the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$.

Let (x_1, y_1, z_1) be a point external to the sphere

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0.$$

Let (x', y', z') be the point of contact of a tangent plane to the sphere which passes through the point (x_1, y_1, z_1),

The equation of the tangent plane at (x', y', z') is

$$xx'+yy'+zz'+u(x+x')+v(y+y')+w(z+z')+d=0.$$

$\because$ it passes through (x_1, y_1, z_1)

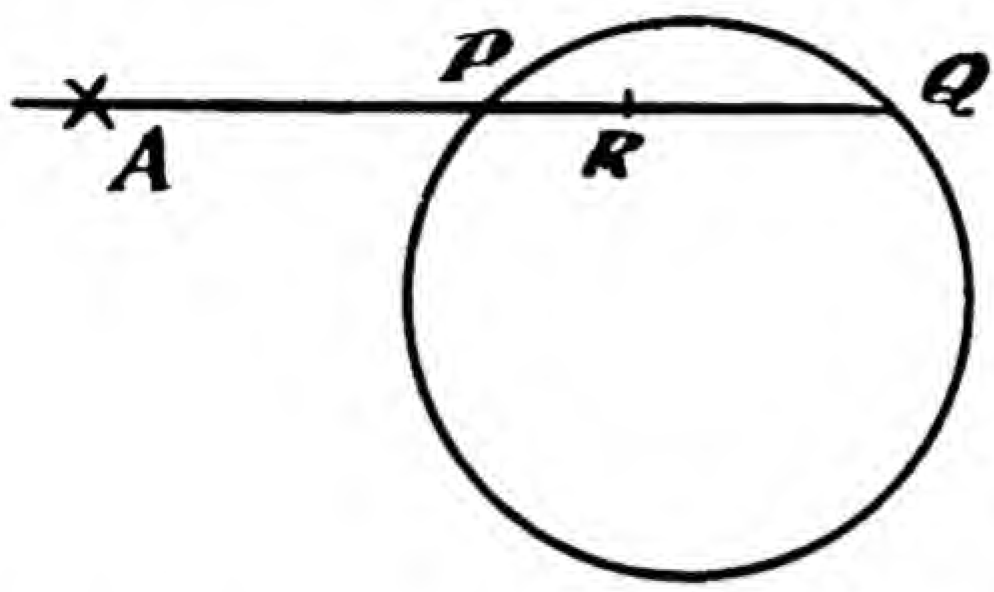
$\therefore \quad x_1x'+y_1y'+z_1z'+u(x_1+x')+v(y_1+y')+w(z_1+z')+d=0.$

$\therefore$ locus of (x', y', z') is

$$xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=0,$$

which is the equation of the plane of contact.

6·15. Polar plane of a given point with respect to a sphere : Def.



If through a given point A, any transversal be drawn to meet a given sphere in P and Q, and if a point R is taken on this line, such that

$$\frac{1}{\text{AP}}+\frac{1}{\text{AQ}}+\frac{2}{\text{AR}},$$

then the locus of the point R, for different positions of the transversal, is called the polar plane of A with respect to the sphere.

6·16. To find the equation of the polar plane of the point (x_1, y_1, z_1) with respect to the sphere

$$\mathbf{x^2+y^2+z^2+2ux+2vy+2wz+d=0.}$$

Let the point A be (x_1, y_1, z_1).

Equations of **any** line through A are

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r, \text{ say.}$$

Let this line meet the given sphere in points P and Q.

Any point on this line is (x_1+lr, y_1+mr, z_1+nr).

If it lies on the given sphere, then

$$(x_1+lr)^2+(y_1+mr)^2+(z_1+nr)^2+2u(x_1+lr)+2v(y_1+mr)$$
$$+2w(z_1+nr)+d=0$$

or,
$$r^2+2r[l(u+x_1)+m(v+y_1)+n(w+z_1)]$$
$$+(x_1{}^2+y_1{}^2+z_1{}^2+2ux_1+2vy_1+2wz_1+d)=0.$$

It is a quadratic equation in r and, therefore, has two roots which correspond to AP and AQ.

$\therefore$ $$\text{AP}+\text{AQ}=-2[l(u+x_1)+m(v+y_1)+n(w+z_1)]$$

and $$\text{AP}.\text{AQ}=x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d.$$

Let R(λ, μ, ν) be a point on APQ such that

$$\frac{1}{\text{AP}}+\frac{1}{\text{AQ}}=\frac{2}{\text{AR}}.$$

Now, $$\frac{2}{\text{AR}}=\frac{1}{\text{AQ}}+\frac{1}{\text{AQ}}=\frac{\text{AP}+\text{AQ}}{\text{AP}.\text{AQ}}$$

$$=\frac{-2[l(u+x_1)+m(v+y_1)+n(w+z_1)]}{x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d}$$

$\therefore$ $$\text{AR}=\frac{-(x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d)}{l(u+x_1)+m(v+y_1)+n(w+z_1)} \quad ...(1)$$

Also, $$\frac{\lambda-x_1}{l}=\frac{\mu-y_1}{m}=\frac{\nu-z_1}{n}=\text{AR}$$

$\therefore$ $$\lambda-x_1=l.\text{AR},$$
$$\mu-y_1=m.\text{AR}$$

and $$\nu-z_1=n.\text{AR} \quad ...(2)$$

Eliminating l, m, n between (1) and (2), we have

$$(\lambda-x_1)(x_1+u)+(\mu-y_1)(v+y_1)+(\nu-z_1)(z_1+w)$$
$$=-(x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d),$$

or, $$\lambda x_1+\mu y_1+\nu z_1+u\lambda+v\mu+w\nu$$
$$=x_1^2+y_1^2+z_1^2+ux_1+vy_1+wz_1-(x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d)$$

$$=-(ux_1+vy_1+wz_1+d),$$

or, $$\lambda x_1+\mu y_1+\nu z_1+u(\lambda+x_1)+v(\mu+y_1)+w(\nu+z_1)+d=0$$

$\therefore$ locus of R (λ, μ, ν) is

$$\mathbf{xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=o},$$

which is the required equation of the polar plane of A with respect to the given sphere.

Note. *The equation of the polar plane of* (x_1, y_1, z_1) *with respect to a sphere is of the* **same form** *as the equation of the tangent plane at* (x_1, y_1, z_1).

Cor. Case of standard sphere.

The equation of the polar plane of the point (x_1, y_1, z_1) with respect to the sphere $x^2+y^2+z^2=a^2$ is

$$\mathbf{xx_1+yy_1+zz_1=a^2}.$$

6·17. Pole of a plane with respect to a sphere : Def.

The point, of which the given plane is the polar plane with respect to the given sphere, is called the pole of the plane with respect to the sphere.

6·18. Conjugate points : Def.

Two points are said to be conjugate with respect to the sphere, if the polar plane of one passes through the other.

6·19. Conjugate planes : Def.

Two planes are said to be conjugate when the pole of one lies on the other.

6·20. Polar lines : Def.

Two lines are said to be polar lines with respect to a sphere when the polar plane of every point of one passes through the other.

EXAMPLES VI (D)

Ex. 1. Reciprocal property.

Prove that if the polar plane of P with respect to a sphere passes through Q, then the polar plane of Q will pass through P.

Sol. Let the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \qquad \dots(1)$$

Let P and Q be the points

$$(x_1, y_1, z_1) \text{ and } (x_2, y_2, z_2).$$

The polar plane of P with respect to (1) is

$$xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=0$$

$\because$ it passes through Q,

$$\therefore \quad x_2x_1+y_2y_1+z_2z_1+u(x_2+x_1)+v(y_2+x_1)+w(z_2+z_1)+d=0 \qquad \dots(2)$$

The polar plane of Q with respect to (1) is

$$xx_2+yy_2+zz_2+u(x+x_2)+v(y+y_2)+w(z+z_2)+d=0.$$

It passes through P in view of (2).

Hence the proposition.

Ex. 2. Prove that the distances of two points from the centre of a sphere are proportional to the distances of each from the polar plane of the other.

Ex. 3. Show that the polar line of

$$x+3=\frac{y+1}{2}=\frac{z-2}{3}$$

with respect to the sphere

$$x^2+y^2+z^2=1$$

is the line

$$\frac{x}{-1}=\frac{7y+3}{11}=\frac{2-7z}{5}. \qquad \textit{(Kashmir, 1959)}$$

SECTION V

ANGLE OF INTERSECTION OF SPHERES. ORTHOGONAL SPHERES

6·21. Angle of intersection of two spheres : Def.

The angle of intersection of two spheres is the angle between the tangent planes to them at a common point of intersection.

Note. *Since the radii to a common point are perpendicular to the tangent planes at that point, therefore the angle of intersection of the spheres is equal to the angle between their radii to the common point.*

6·22. Orthogonal spheres : Def.

If the angle of intersection of two spheres is a right angle, then the spheres are said to be orthogonal or said to cut orthogonally.

6·23. To find the condition that the spheres

$$\mathbf{x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0}$$

and

$$\mathbf{x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0}$$

intersect orthogonally.

The given spheres are

$$x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0 \qquad ...(1)$$

and

$$x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0 \qquad ...(2)$$

Let C_1 and C_2 be their centres and P be a common point of intersection.

$\because$ (1) and (2) are orthogonal,

$\therefore$
$$C_1C_2^2=C_1P^2+C_2P^2,$$

or
$$(u_1-u_2)^2+(v_1-v_2)^2+(w_1-w_2)^2=(u_1^2+v_1^2+w_1^2-d_1)+(u_2^2+v_2^2+w_2^2-d_2),$$

or
$$\mathbf{2(u_1u_2+v_1v_2+w_1w_2)=d_1+d_2},$$

which is the required condition.

EXAMPLES VI (E)

Type I. Ex. 1. Prove that the spheres

$$\mathbf{x^2+y^2+z^2+6y+2z+8=0}$$

and

$$\mathbf{x^2+y^2+z^2+6x+8y+4z+20=0}$$

are orthogonal.

(Punjab, 1957 ; Punjab, B.Sc., 1959 S)

Sol.

Here $u_1=0,\ v_1=3,\quad w_1=1$ and $d_1=8$.

$u_2=3,\ v_2=4,\quad w_2=2$ and $d_2=20$.

Now, $2(u_1u_2+v_1v_2+w_1w_2)=2(0+12+2)=28.$

Also, $d_1+d_2=8+20=28$.

$\therefore$ $2(u_1u_2+v_1v_2+w_1w_2)=d_1+d_2$.

$\therefore$ by Art, 6·23, the given spheres intersect orthogonally.

Ex. 2. Prove that a sphere which cuts the two spheres $S=0$, $S'=0$ orthogonally will also cut $lS+mS'=0$ orthogonally. *(Gauhati, 1960)*

Ex. 3. Show that every sphere through the circle

$$x^2+y^2-2ax+r^2=0,\ z=0$$

cuts orthogonally every sphere through the circle

$$x^2+z^2=r^2,\ y=0.$$

Type II. Ex. 1. Find the equation of a sphere which touches the plane $3x+2y-z+2=0$ at $(1, -2, 1)$ and cuts orthogonally the sphere

$$\mathbf{x^2+y^2+z^2-4x+6y+4=0}$$

(Karnatak Engg., 1961 ; Punjab, 1959)

Sol. $\because$ the required sphere touches the given plane at $(1, -2, 1)$,

$\therefore$ the centre of the required sphere lies on the normal to the given plane through the point $(1, -2, 1)$. Equations of the normal to the given plane through $(1, -2, 1)$ are

$$\frac{x-1}{3}=\frac{y+2}{2}=\frac{z-1}{-1}=r, \text{ say.}$$

$\therefore$ coordinates of the centre of the required sphere can be written as

$$(1+3r,\ -2+2r,\ 1-r), \text{ and}$$

the radius of the required sphere is $r\sqrt{14}$.

$\because$ the required sphere intersects the sphere

$$x^2+y^2+z^2-4x+6y+4=0 \text{ orthogonally,}$$

$\therefore$ (distance between their centres)^2

= the sum of the squares of their radii,

or, $$(3r+1-2)^2+(2r-2+3)^2+(-r+1-0)^2=14r^2+9,$$

or, $$(3r-1)^2+(2r+1)^2+(r-1)^2=14r^2+9,$$

or, $$r=-\frac{3}{2}.$$

$\therefore$ Centre of the required sphere is

$$\left(-\frac{7}{2}, -5, \frac{5}{2}\right)$$

and its radius

$$=\frac{3}{2}\sqrt{14}.$$

$\therefore$ its equation is

$$\left(x+\frac{7}{2}\right)^2+(y+5)^2+\left(z-\frac{5}{2}\right)^2=\frac{63}{2},$$

or, $$x^2+y^2+z^2+7x+10y-5z+12=0.$$

Ex. 2. Find the equation of the sphere which cuts orthogonally each of the spheres

$$x^2+y^2+z^2=a^2+b^2+c^2,$$
$$x^2+y^2+z^2+2ax=a^2,$$
$$x^2+y^2+z^2+2by=b^2,$$
$$x^2+y^2+z^2+2cz=c^2.$$

$\left[\text{Ans. } x^2+y^2+z^2+\dfrac{b^2+c^2}{a}x+\dfrac{c^2+a^2}{b}y+\dfrac{a^2+b^2}{c}z+a^2+b^2+c^2=0.\right]$

SECTION VI

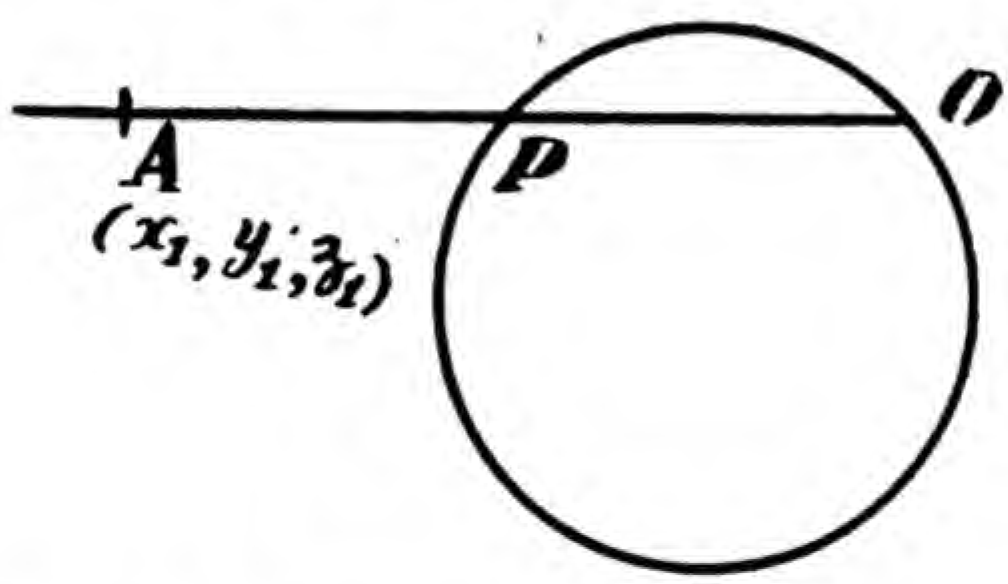
POWER OF A POINT WITH RESPECT TO A SPHERE. RADICAL PLANE. RADICAL LINE. RADICAL CENTRE. COAXAL SPHERES

6·24. To show that if any secant through a point A meets a given sphere in P and Q, then AP. AQ is constant.

Proof. Let the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \qquad \ldots(1)$$

Let the point A be (x_1, y_1, z_1).



Let a line through A meet (1) in P and Q.

Equations of the line APQ are

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r,$$

say, where $l, m, n,$ are the actual direction cosines.

Any point on this line is (x_1+lr, y_1+mr, z_1+nr).

If it is lies on (1), then

$$(x_1+lr)^2+(y_1+mr)^2+(z_1+nr)^2+2u(x_1+lr)+2v(y_1+mr)$$
$$+2w(z_1+nr)+d=0,$$

or $$r^2(l^2+m^2+n^2)+2r[l(u+x_1)+m(v+y_1)+n(w+z_1)]$$
$$+(x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d)=0,$$

or, $$r^2+2r[l(u+x_1)+m(v+y_1)+n(w+z_1)]$$
$$+(x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d)=0.$$

This is a quadratic equation in r. It has two roots AP and AQ.

$$\therefore \qquad AP.AQ=x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d,$$

which is constant as the result is independent of l, m, n.

This proves the proposition.

6·25. Power of a point with respect to a sphere : Def.

If a straight line thorugh a point A meets the sphere in points P and Q, then the constant quantity AP.AQ is called the power of the point A with respect to the sphere.

Note. *The power of the point* (x_1, y_1, z_1) *with respect to the sphere* $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ *is* $\mathbf{x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d}$.

6·26. Radical plane of two spheres : Def.

The radical plane of two spheres is the locus of a moving point whose powers with respect to the two spheres are equal.

6·27. Equations of the radical plane of two spheres. To find the equations of the radical plane of the spheres

$$\mathbf{x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=o}$$

and $$\mathbf{x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=o.}$$

The given spheres are

$$x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0 \qquad \ldots(1)$$

and $$x^2+y^2+z^2+2u_2x+2v_2x+2w_2z+d_2=0 \qquad \ldots(2)$$

Let (λ, μ, ν) be **any** point on the radical plane of (1) and (2).

$\therefore$ by def., the power of (λ, μ, ν) with respect to (1)=the power of (λ, μ, ν) with respect to (2),

or, $$\lambda^2+\mu^2+\nu^2+2u_1\lambda+2v_1\mu+2w_1\nu+d_1=\lambda^2+\mu^2+\nu^2+2u_2\lambda$$
$$+2v_2\mu+2w_2\nu+d_2,$$

or, $$2\lambda(u_1-u_2)+2\mu(v_1-v_2)+2\nu(w_1-w_2)+d_1-d_2=0.$$

$\therefore$ locus of (λ, μ, ν) is

$$\mathbf{2x(u_1-u_2)+2y(v_1-v_2)+2z(w_1-w_2)+d_1-d_2=o,}$$

which is the required equation of the radical plane of (1) and (2).

Aid to memory. The equation of the radical plane is obtained by subtracting the equation of one sphere from that of the other, after making the coefficients of x^2, y^2, z^2 in each unity and R.H.S. of each zero.

6·28. Properties of radical plane.

(i) To show that the radical plane of two spheres is at right angles to the line joining the centres.

Proof. Let the spheres be

$$x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0 \qquad \ldots(1)$$

and

$$x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0 \qquad \ldots(2)$$

Equation of the radical plane of (1) and (2) is

$$2x(u_1-u_2)+2y(v_1-v_2)+2z(w_1-w_2)+d_1-d_2=0 \qquad \ldots(3)$$

Direction cosines of the normal to (3) are proportional to

$$u_1-u_2,\ v_1-v_2,\ w_1-w_2.$$

Also, the direction cosines of the line joining the centres $(-u_1, -v_1, -w_1)$ and $(-u_2, -v_2, -w_2)$ are proportional to u_1-u_2, v_1-v_2, w_1-w_2.

$\therefore$ the normal to (3) is parallel to the line of centres.

$\therefore$ the radical plane (3) is perpendicular to the line of centres. This proves the proposition.

(ii) To show that the radical plane of two spheres passes through their points of intersection.

Proof. Let the spheres be

$$x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0 \qquad \ldots(1)$$

and

$$x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0 \qquad \ldots(2)$$

The equation of the radical plane of (1) and (2) is

$$2x(u_1-u_2)+2y(v_1-v_2)+2z(w_1-w_2)+d_1-d_2=0 \qquad \ldots(3)$$

Now, (3) is satisfied by the coordinates of the points which satisfy both (1) and (2).

$\therefore$ the points of intersection of (1) and (2) lie on (3).

Hence the radical plane of the two spheres passes through their points of intersection.

This proves the proposition.

6·29. Radical line of three spheres.

To prove that the radical planes of three spheres, taken two by two, pass through a line.

Proof. Let the three spheres be given by the equations

$$S_1=0,\ S_2=0,\ S_3=0.$$

The radical plane of the spheres $S_1=0$ and $S_2=0$ is

$$S_1=S_2 \qquad \ldots(1)$$

The radical plane of the spheres $S_1=0$ and $S_3=0$ is

$$S_1=S_3 \qquad \ldots(2)$$

The radical plane of the spheres $S_2=0$ and $S_3=0$ is

$$S_2=S_3 \qquad \ldots(3)$$

$\therefore$ the radical planes of the spheres $S_1=0$, $S_2=0$, $S_3=0$ pass through the line $S_1=S_2=S_3$.

This proves the proposition.

6·30. Radical line: Def.

The three radical planes of three spheres, taken in pairs, pass through a line called the **radical line** of the three spheres.

6·31. To prove that the radical planes of four spheres, taken in pairs, pass through a point.

Proof. Let $S_1=0$, $S_2=0$, $S_3=0$, $S_4=0$ be the four spheres.

The radical planes of the spheres taken in pairs are

$$S_1-S_2=0,\ S_1-S_3=0,\ S_1-S_4=0,\ S_2-S_3=0,\ S_2-S_4=0$$

and

$$S_3-S_4=0.$$

$\therefore$ these planes meet in a point $S_1=S_3=S_2=S_4$.

This proves the proposition.

6·32. Radical Centre: Def.

The radical planes of four spheres taken two by two pass through one point, called the **radical centre** of the four spheres.

6·33. Simplest form of the equation of two spheres.

To show that the equations of any two spheres can be put in the form

$$\mathbf{x^2+y^2+z^2+2\lambda_1 x+d=o}$$

and

$$\mathbf{x^2+y^2+z^2+2\lambda_2 x+d=o.}$$

Proof. Let the equations of the spheres be

$$x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0 \qquad ...(1)$$

and
$$x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0 \qquad ...(2)$$

(i) Take the line of centres as the x-axis.

$\because$ the centres $(-u_1, -v_1, -w_1)$ and $(-u_2, -v_2, -w_2)$ lie on the x-axis $(y=0, z=0)$,

$\therefore$ $v_1=0$, $w_1=0$, $v_2=0$ and $w_2=0$.

$\therefore$ the equations of the spheres become

$$x^2+y^2+z^2+2u_1x+d_1=0$$

and
$$x^2+y^2+z^2+2u_2x+d_2=0. \qquad ...(3)$$

***(ii)* Take the radical plane as yz-plane.**

The equation of the radical plane is

$$2x(u_1-u_2)+(d_1-d_2)=0 \qquad ...(4)$$

$\because$ (4) is the same as the yz-plane, *i.e.*, $x=0$,

$\because$ comparing coefficient of like terms, we have

$$\frac{(u_1-u_2)}{1}=\frac{d_1-d_2}{0},$$

or,
$$d_1-d_2=0.$$

$\therefore$
$$d_1=d_2=d, \text{ say}.$$

$\therefore$ equations (3) become

$$x^2+y^2+z^2+2u_1x+d=0$$

and
$$x^2+y^2+z^2+2u_2x+d=0.$$

Hence the equations of two spheres can be put in the form

$$x^2+y^2+z^2+2\lambda_1x+d=0$$

and
$$x^2+y^2+z^2+2\lambda_2x+d=0.$$

6·34. Coaxial system of spheres : Def.

A system of spheres such that any two of them have the same radical plane is called a **coaxal system of spheres.**

6.35. Standard equation of a coaxal system.

To show that $x^2+y^2+z^2+2\lambda x+d=0$, reprents a system of coaxal spheres, λ being a parameter.

Proof. The given equation is

$$x^2+y^2+z^2+2\lambda x+d=0 \qquad ...(1)$$

λ being a parameter.

$\because$ λ is variable,

$\therefore$ (1) represents a system of spheres.

Let the equations of two members of (1) be

$$x^2+y^2+z^2+2\lambda_1x+d=0$$
$$x^2+y^2+z^2+2\lambda_2x+d=0.$$

The equation of their radical plane is

$$2(\lambda_1-\lambda_2)x=0,$$

or, $$x=0,$$

which is indepentent of λ_1 and λ_2 and is therefore the same for every two spheres of the system.

Hence, by def.,

$$x^2+y^2+z^2+2\lambda x+d=0,$$

where λ is variable and d is constant represents a system of coaxial spheres. This proves the proposition.

Aid to memory. The equation

$$\mathbf{x^2+y^2+z^2+2\lambda x+d=0},$$

where λ is variable and d is constant represents a system of coaxal spheres.

6·36. Types of coaxal spheres.

To show that the members of the coaxal system

$$\mathbf{x^2+y^2+z^2+2\lambda x+d=0},$$

intersect one another, touch one another or do not intersect one another according as $\mathbf{d \lesseqgtr 0}$.

Proof. Let the two members of the given system of coaxal spheres be

$$x^2+y^2+z^2+2\lambda_1x+d=0 \qquad \ldots(1)$$

and $$x^2+y^2+z^2+2\lambda_2x+d=0 \qquad \ldots(2)$$

They intersect where,

$$2(\lambda_1-\lambda_2)\,x=0,$$

or $$x=0.^*$$

Substituting $x=0$ in (1), we have

$$y^2+z^2+d=0,$$

or, $$y^2+z^2=-d \qquad \ldots(3)$$

which represents a circle in yz-plane of radius $\sqrt{-d}$. ...(4)

Hence the spheres (1) and (2) intersect, touch, or do not intersect, according as the circle (3) is a real circle, a point circle, or an imaginary circle,

i.e., according as its radius is real, zero, or imaginary,

---

* This is obtained on subtracting (2) from (1).

i.e., according as d is negative, zero, or positive,

i.e., according as $d \lesseqgtr 0$. This proves the proposition.

6·37. Limiting points : Def.

The limiting points of a coaxal system are the spheres of zero radius belonging to the system.

6·38. To find the limiting points of a coaxal system of spheres whose equation is

$$\mathbf{x^2+y^2+z^2+2\lambda x+d=0.}$$

Let the equation of any member of the coaxal system be

$$x^2+y^2+z^2+2\lambda x+d=0.$$

Its centre is $(-\lambda, 0, 0)$ and its radius is $\sqrt{\lambda^2-d}$.

For limiting points, radius$=0$.

$$\therefore \qquad \lambda^2=d,$$

or,
$$\lambda=\pm\sqrt{d}.$$

$\therefore$ the two limiting points are $(\pm\sqrt{d}, 0, 0)$.

Note. *The limiting points are real, coincident or imaginary according as $d \gtreqless 0$, i.e., according as the the coaxal system is non-intersecting, touching or intersecting.*

6·39. If $S_1=0$, $S_2=0$ be two spheres, to show that $S_1+\lambda S_2=0$ represents a system of coaxal spheres, λ being a parameter.

Proof: $\because$ λ is a parameter,

$$\therefore \qquad S_1+\lambda S_2=0$$

represents a system of spheres. Let its two members be

$$S_1+\lambda_1 S_2=0$$

and
$$S_2+\lambda_2 S_2=0 \qquad \ldots(1)$$

The equation of their radical plane is

$$(1+\lambda_2)(S_1+\lambda_1 S_2)-(1+\lambda_1)(S_1+\lambda_2 S_2)=0,^*$$

or
$$(\lambda_2-\lambda_1)(S_1-S_2)=0,$$

or.
$$S_1-S_2=0.$$

---

* Making coefficients of x^2, y^2, z^2 each unity, the equations (1) become

$$\frac{S_1+\lambda_1 S_2}{1+\lambda_1}=0 \quad \text{and} \quad \frac{S_1+\lambda_2 S_2}{1+\lambda_2}=0.$$

The radical plane of these is

$$\frac{S_1+\lambda_1 S_2}{1+\lambda_1}-\frac{S_1+\lambda_2 S_2}{1+\lambda_2}=0,$$

or, $(1+\lambda_2)(S_1+\lambda_1 S_2)-(1+\lambda_1)(S_1+\lambda_2 S_2)=0.$

$\because$ this is independent of λ_1 and λ_2,

$\therefore$ we conclude that every two members of the system have the same radical plane.

Hence $S_1+\lambda S_2=0$ represents (by def.) a system of coaxal spheres. This proves the proposition.

EXAMPLES VI (F)

Type I. (Problems on radical plane, radical line and radical centre.)

Ex. 1. Three spheres of radii r_1, r_2, r_3 have their centres at A, B, C at the points (a, 0, 0), (0, b, 0), (0, 0, c) and $r_1^2+r_2^2+r_3^2=a^2+b^2+c^2$. A fourth sphere passes through the origin and the points A, B, C. Show that the radical centre of the four spheres lies on the plane $ax+by+cz=0$.

Sol. The equations of the four given spheres are

$$(x-a)^2+y^2+z^2=r_1^2 \quad ...(1)$$

$$x^2+(y-b)^2+z^2=r_2^2 \quad ...(2)$$

$$x^2+y^2+(z-c)^2=r_3^2 \quad ...(3)$$

and $$x^2+y^2+z^2-ax-by-cz=0 \quad ...(4)$$

Radical plane of (1) and (4) is

$$-ax+by+cz+a^2-r_1^2=0 \quad ...(5)$$

Radical plane of (2) and (4) is

$$ax-by+cz+b^2-r_2^2=0 \quad ...(6)$$

Radical plane of (3) and (4) is

$$ax+by-cz+c^2-r_3^2=0 \quad ...(7)$$

The radical centre is the point common to (5), (6) and (7).

$\therefore$ it lies on the plane

$$(-ax+by+cz+a^2-r_1^2)+(ax-by+cz+b^2-r_2^2)+(ax+by-cz+c^2-r_3^2)=0,$$

or, $$ax+by+cz+(a^2+b^2+c^2)-(r_1^2+r_2^2+r_3^2)=0,$$

or, $$ax+by+cz=0, \text{ because}$$

$$a^2+b^2+c^2=r_1^2+r_2^2+r_3^2.$$

Ex. 2. Find the equation of the radical line of the spheres

$$x^2+y^2+z^2+2x+2y+2z+2=0, \quad x^2+y^2+z^2+4y=0$$

and $$x^2+y^2+z^2+3x-2y+8z+6=0.$$

[**Ans.** $x-y+z+1=0,\ 3x-6y+8z+6=0$]

Ex. 3. A and B are two fixed points and P moves so that $PA=n.PB$; show that the locus of P is a sphere. Show also that all such spheres, for different values of n, have a common radical plane.

Type II. Ex. 1. Show that the spheres which cut two given spheres along great circles all pass through two fixed points.

Sol. Let the two given spheres be

$$x^2+y^2+z^2+2\lambda x+d=0 \quad ...(1)$$

and $$x^2+y^2+z^2+2\mu x+d=0 \quad ...(2)$$

Let any other sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+c=0 \qquad \text{...(3)}$$

The spheres (3) will cut sphere (1) in a great circle, if the centre of (1) lies on the radical plane of (1) and (3).

Now, the centre of (1) is $(-\lambda, 0, 0)$ and the radical plane of (1) and (3) is

$$2ux+2vy+2wz+c-2\lambda x-d=0.$$

$\therefore$ (3) will cut (1) in a great circle if

$$-2u\lambda+c+2\lambda^2-d=0,$$

or,

$$2\lambda^2-2u\lambda+c-d=0 \qquad \text{...(4)}$$

Similarly, (3) cuts (2) in a great circle, if

$$2\mu^2-2u\mu+c-d=0 \qquad \text{...(5)}$$

Solving (4) and (5), we have

$$u=\lambda+\mu \quad \text{and} \quad c=2\lambda\mu+d.$$

$\therefore$ (3) becomes

$$x^2+y^2+z^2+2(\lambda+\mu)x+2vy+2wz+2\lambda\mu+d=0 \qquad \text{...(6)}$$

All such spheres clearly pass through the locus given by the equations

$$y=0, \quad z=0$$

and

$$x^2+y^2+z^2+2(\lambda+\mu)x+d+2\lambda\mu=0.$$

which determine two fixed points on the x-axis.

Ex. 2. Show that, in general, there are two spheres of a coaxal system which touch a given plane. Find the equations to the two spheres of the system

$$x^2+y^2+z^2+2\lambda x+3=0$$

which touch the plane

$$x+2y+2z+4=0.$$

[**Ans.** $x^2+y^2+z^2+4x+3=0,\ 2(x^2+y^2+z^2)-13x+3=0.$]

Type III. Ex. 1. Find the limiting points of the system defined by the spheres $\mathbf{x^2+y^2+z^2+4x-2y+2z+6=0}$

and $\mathbf{x^2+y^2+z^2+2x-4y-2z+6=0.}$

Sol. The radical plane of the given spheres is

$$2x+2y+4z=0,$$

or,

$$x+y+2z=0.$$

Equation of any sphere belonging to the system defined by the given spheres is

$$x^2+y^2+z^2+4x-2y+2z+6+\lambda\,(x+y+2z)=0.$$

The centre of this sphere is

$$\left(-\frac{4+\lambda}{2},\ -\frac{\lambda-2}{2},\ -\frac{2+2\lambda}{2}\right) \qquad \text{...(1)}$$

$$\text{Its radius}=\sqrt{\left(\frac{4x+\lambda}{2}\right)^2+\left(\frac{\lambda-2}{}\right)^2+\left(\frac{2+2\lambda}{2}\right)^2-6}$$

$$=\left(\frac{3\lambda^2+6\lambda}{2}\right)^{1/2}.$$

Now, the limiting points are spheres of zero radius belonging to the system.

$$\therefore \qquad 3\lambda^2+6\lambda=0, \quad \text{or,} \quad \lambda=0,\ -2.$$

$\therefore$ (3) gives the limiting points as

$$(-2, 1, -1) \quad \text{and} \quad (-1, 2, 1).$$

Ex. 2. Find the limiting points of the coaxal system of spheres determined by

$$x^2+y^2+z^2+3x-3y+6=0$$

and

$$x^2+y^2+z^2-6y-6z+6=0.$$

[**Ans.** $(-2, 1, -1), (-1, 2, 1)$.]

Type IV. Ex. 1. Find the equation of the sphere belonging to the coaxal system defined by

$$\mathbf{x^2+y^2+z^2-2ax-2ay-2az+4a^2=0}$$

and

$$\mathbf{x^2+y^2+z^2-4ax-4ay+4a^2=0}$$

and which cuts the sphere

$$\mathbf{x^2+y^2+z^2+2ax=0} \textbf{ orthogonally.}$$

Sol. The radical plane of the given spheres is

$$2ax+2ay-2az=0, \quad \text{or,} \quad x+y-z=0.$$

$\therefore$ equation of any sphere belonging to the system defined by the given spheres is

$$x^2+y^2+z^2-2ax-2ay-2az+4a^2+\lambda(x+y-z)=0 \quad \text{.. (1)}$$

or

$$x^2+y^2+z^2+(\lambda-2a)x+(\lambda-2a)y-(\lambda+2a)z+4a^2=0.$$

This sphere cuts $x^2+y^2+z^2+2ax=0$ orthogonally if

$$a(\lambda-2a)=4a^2, \quad \text{or if} \quad \lambda=6a.$$

$\therefore$ (1) gives $x^2+y^2+z^2+4ax+4ay-8az+4a^2=0$,

which is the required equation.

Ex. 2. Find the equation of the sphere belonging to the coaxal system defined by the spheres

$$x^2+y^2+z^2+3x-3y+2z=0,$$
$$x^2+y^2+z^2+2x-y-z+10=0,$$

and which passes through (0, 1, 2).

[**Ans.** $x^2+y^2+z^2+4x-5y+5z-10=0$.]

Ex. 3. Find the epuations of the spheres of the coaxal system whose limiting points are $(-1, 2, 1)$ and $(-2, 1, -1)$ and which touches the plane

$$2x+3y+6z+7=0.$$

[**Ans.** $x^2+y^2+z^2+4x-2y+2z+6=0$,
$5(x^2+y^2+z^2)-568x-598y-1166z+30=0$.]

MISCELLANEOUS (REVISION) EXAMPLES ON CHAPTER VI

1. If the axes are rectangular, show that the locus of the centre of a circle of radius a which always intersects them is

$$\mathbf{x\sqrt{a^2-y^2-z^2}+y\sqrt{a^2-z^2-x^2}+z\sqrt{a^2-x^2-y^2}=a^2.}$$

(Raj., 1952)

Sol. Let $P(\alpha, \beta, \gamma)$ be the centre of a circle of radius a which intersects the axes in A, B and C respectively.

Let OA be l, OB be m and OC be n.

$\therefore$ coordinates of A, B, C are respectively (l, o, o), (o, m, o) and (o, o, n)

Now, $$PA=PB=PC=a,$$

or $$PA^2=PB^2=PC^2=a^2$$

or, $$(l-\alpha)^2+\beta^2+\gamma^2=\alpha^2+(m-\beta)^2+\gamma^2=\alpha^2+\beta^2+(n-\gamma)^2=(a^2).$$

$$\therefore \quad l=\sqrt{a^2-\beta^2-\gamma^2}+\alpha,\ m=\sqrt{a^2-\gamma^2-\alpha^2}+\beta,\ n=\sqrt{a^2-\alpha^2-\beta^2}+\gamma \quad \ldots(1)$$

The equation of the plane of this circle is

$$\frac{x}{l}+\frac{y}{m}+\frac{z}{n}=1.$$

$\because$ the centre lies on it,

$$\frac{\alpha}{l}+\frac{\beta}{m}+\frac{\gamma}{n}=1. \quad \ldots(2)$$

From (1) and (2), on eliminating l, m, n, we have

$$\frac{\alpha}{\sqrt{a^2-\beta^2-\gamma^2}+\alpha}+\frac{\beta}{\sqrt{a^2-\gamma^2-\alpha^2}+\beta}+\frac{\gamma}{\sqrt{a^2-\alpha^2-\beta^2}+\gamma}=1.$$

On rationalising, we have

$$\alpha\sqrt{a^2-\beta^2-\gamma^2}-\alpha^2+\beta\sqrt{a^2-\gamma^2-\alpha^2}-\beta^2+\gamma\sqrt{a^2-\alpha^2-\beta^2}-\gamma^2$$
$$=a^2-(\alpha^2+\beta^2+\gamma^2),$$

or, $$\alpha\sqrt{a^2-\beta^2-\gamma^2}+\beta\sqrt{a^2-\gamma^2-\alpha^2}+\gamma\sqrt{a^2-\alpha^2-\beta^2}=a^2$$

$\therefore$ locus of (α, β, γ) is

$$x\sqrt{a^2-y^2-z^2}+y\sqrt{a^2-z^2-x^2}+z\sqrt{a^2-x^2-y^2}=a^2.$$

2. A variable plane is parallel to the given plane

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

and meets the axes in A, B, C respectively. Prove that the circle ABC lies on the cone

$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{a}{c}+\frac{c}{a}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0,$$

(*Delhi Hons., 1959 ; Punjab, 1957 ; Karnatak, 1961 ; Vikram Engg., 1960 ; Raj., 1954 ; Banaras, 1952; Vikram, 1962*)

Sol. Equation of any plane parallel to the given plane is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=k \quad \ldots(1)$$

$\therefore$ The intercepts cut off from the axes by (1) are ka, kb, kc respectively.

The equation of the sphere OABC is

$$x^2+y^2+z^2-akx-bky-ckz=0,$$

or, $$x^2+y^2+z^2-k(ax+by+cz)=0, \quad \ldots(2)$$

The circle ABC in question is the curve of intersection of (1) and (2).

The locus of the circle ABC will be obtained by eliminating λ between (1) and (2).

$\therefore$ the required locus of the $\odot$ ABC is

$$(x^2+y^2+z^2)-\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)(ax+by+cz)=0,$$

or,
$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0,$$

which is a cone.

3. P is a variable point on a given line and A, B, C are its projections on the axes. Show that the sphere OABC passes through a fixed circle.
(*A.M.I.E., Nov., 1959 ; Baroda, 1954 ; Kashmir, 1958*)

4. **Prove that the centres of spheres which touch the lines $y=mx$, $z=c$; $y=-mx$, $z=-c$, lie upon the conicoid $mxy+cz(1+m^2)=0$.**
(*Delhi Hons., 1953 ; Karnatak, 1961 ; Punjab, 1955*)

Sol. Let the sphere be $x^2+y^2+z^2+2ux+2vy+2wz+d=0$. ...(1)

It meets $y=mx$, $z=c$ ...(2)

where
$$x^2+m^2x^2+c^2+2ux+2vmx+2wc+d=0,$$

or,
$$x^2(1+m^2)+2(u+vm)x+(c^2+2wc+d)=0, \quad ...(3)$$

$\because$ (2) touches (1), $\therefore$ (3) has equal roots, the condition for which is

$$(u+vm)^2=(1+m^2)(c^2+2wc+d) \quad ...(4)$$

on cancelling 4 from both the sides.

Similarly, since the line $y=-mx$, $z=-c$ touches (1),

$\therefore$
$$(u-vm)^2=(1+m^2)(c^2-2wc+d) \quad ...(5)$$

Subtracting (5) from (4), we have

$$uvm=2wc(1+m^2) \quad ...(6)$$

Let the centre of (1) be (λ, μ, ν).

$\therefore$
$$\lambda=-u,\ \mu=-v,\ \nu=-w \quad ...(7)$$

Eliminating u, v, w between (6) and (7), we have

$$\lambda\mu m=-\nu c(1+m^2).$$

$\therefore$ locus of (λ, μ, ν) is $xym+zc(1+m^2)=0$.

Aliter.

Let p_1, p_2 be perpendicular distances of the centre of the sphere, *viz.*, (λ, μ, ν) from the lines $y=mx$, $z=c$ and $y=-mx$, $z=-c$ respectively.

$\therefore$
$$p_1^2=(\nu-c)^2+\frac{(\mu-m\lambda)^2}{1+m^2}$$

and
$$p_2^2=(\nu+c)^2+\frac{(\mu+m\lambda)^2}{1+m^2}$$

$\because$
$$p_1=p_2, \quad \therefore\ p_1^2=p_2^2,$$

or,
$$(\nu-c)^2+\frac{(\mu-m\lambda)^2}{1+m^2}=(\nu+c)^2+\frac{(\mu+m\lambda)^2}{1+m^2}$$

or,
$$\frac{1}{m^2+1}[(\mu+m\lambda)^2-(\mu-m\lambda)^2]+(\nu+c)^2-(\nu-c)^2=0.$$

or, $$m\lambda\mu+c\nu(1+m^2)=0.$$

$\therefore$ locus of (λ, μ, ν) is $mxy+cz(1+m^2)=0$.

5. **Two spheres of radii r_1 and r_2 cut orthogonally. Prove that the radius of the common circle is $r_1r_2/\sqrt{r_1^2+r_2^2}$.**

(Delhi Hons., 1962 ; Punjob 1956 ; Punjab B.Sc., 1962 ; Raj., 1961 ; Bihar, 1961)

Sol. Let the common circle be $x^2+y^4=a^2$, $z=0$ ...(1)

The equation of the sphere passing through this circle is

$$x^2+y^2+z^2-a^2+2kz=0 \quad ...(2)$$

Let two such spheres be $x^2+y^2+z^2-a^2+2k_1z=0$ ...(3)

and $$x^2+y^2+z^2-a^2+2k_2z=0 \quad ...(4)$$

$\because$ (3) and (4) intersect orthogonally,

$\therefore$ $$2k_1k_2=2a^2, \quad \text{or,} \quad k_1k_2=a^2 \quad ...(5)$$

Let r_1 and r_2 be the radii of (3) and (4) respectively,

$\therefore$ $$r_1^2=k_1^2+a^2 \quad \text{and} \quad r_2^2=k_2^2+a^2.$$

$\therefore$ $$k_1^2k_2^2=(r_1^2-a^2)(r_2^2-a^2),$$

or, $$a^4=r_1^2r_2^2-a^2(r_1^2+r_2^2)+a^4, \text{ using (5).}$$

$\therefore$ $$a^2=\frac{r_1^2r_2^2}{(r_1^2+r_2^2)}, \text{ or, } a=r_1r_2/\sqrt{r_1^2+r_2^2}.$$

Aliter.

Let the two spheres be $x^2+y^2+z^2=r_1^2$ ...(1)

and $$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad ...(2)$$

Let the radius of (2) be r_2.

$\therefore$ $$r_2^2=u^2+v^2+w^2-d \quad ...(3)$$

$\because$ (1) and (2) intersect orthogonally,

$\therefore$ $$d_1-r_1^2=0 \quad ...(4)$$

The plane of the common circle is

$$(x^2+y^2+z^2+2ux+2vy+2wz+d)-(x^2+y^2+z^2-r_1^2)=0,$$

or, $$ux+vy+wz+r_1^2=0. \quad ...(5), \text{ using (4)}$$

Let p be the length of the perpendicular from (0, 0, 0) on (5).

$\therefore$ $$p^2=r_1^4/(u^2+v^2+w^2)=\frac{r_1^4}{r_2^2+d}, \text{ using (3)}$$

$$=r_1^4/(r_2^2+r_1^2), \text{ using (4).}$$

Let a be the radius of the common circle.

$\therefore$ $$a^2=r_1^2-p^2=r_1^2+\frac{r_1^4}{r_1^2+r_2^2}=\frac{r_1^2r_2^2}{r_1^2+r_2^2}.$$

$\therefore$ $$a=r_1r_2/\sqrt{r_1^2+r_2^2}.$$

7

The Cone

SECTION I

DEFINITION AND EQUATION OF THE CONE

7·1. Cone : Def.

A cone is a surface generated by a straight line (called th **generator** of the cone) which passes through a fixed point (called the **vertex** of the cone) and satisfies some other condition, for instance it may intersect a given curve (called the **guiding curve**) or touch a given surface or make a certain angle with a straight line through the fixed point.

Note. *A cone whose equation is of the second degree, is called a* **quadric cone.**

7·2. Homogeneous equation : Def.

An equation $f(x, y, z)=0$ is said to be homogeneous in x, y and z if $f(tx, ty, tz)=0$ for all values of t. For example

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$$

is a homogeneous equation in x, y and z.*

7·3. Equation of the cone with vertex at the origin. To prove that the equation of a cone with vertex at the origin is homogeneous in x, y, z.

(Nagpur T.D.C., 1962)

Proof. Let $f(x, y, z)=0$ ...(1)

be the equation of the cone with vertex at the origin.

---

*Let $f(x, y, z) \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$.

$\therefore \quad f(tx, ty, tz) \equiv a(tx)^2+b(ty)^2+c(tz)^2+2f(ty)(tz)+2g(tz)(tx)+2h(tx)(ty)$
$\qquad = t^2(ax^2+by^2+cz^2+2fyz+2gzx+2hxy)=0.$

(*i*) Let P (x_1, y_1, z_1) be **any** point on the cone.

$\therefore \qquad f(x_1, y_1, z_1)=0 \qquad$ ...(2)

(*ii*) Also, the equation of the generator OP is

$$\frac{x}{x_1}=\frac{y}{y_1}=\frac{z}{z_1}$$

Any point on OP is (tx_1, ty_1, tz_1). ...(3)

(*iii*) $\because$ The generator completely lies on the cone,

$\therefore$ the point (tx_1, ty_1, tz_1) must lie on the cone.

$\therefore$ $f(tx_1, ty_1, tz_1)=0$ for all values of t. ...(4)

(*iv*) From (2) and (4), the equation $f(x_1, y_1, z_1)=0$ is homogeneous in (x_1, y_1, z_1) (Def. Art. 7·2).

(*v*) $\therefore$ $f(x, y, z)=0$ is homogeneous in x, y, z. This proves the proposition.

Aliter.

Let $ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux$
$+2vy+2wz+d=0$ ...(1)

represent a cone whose vertex is at the origin.

We shall prove that $u=0=v=w=d$.

Let $P(x_1, y_1, z_1)$ be **any** point on (1).

The equations of OP are

$$\frac{x}{x_1}=\frac{y}{y_1}=\frac{z}{z_1}.$$

Any point on OP is (tx_1, ty_1, tz_1).

$\because$ OP is a generator of the cone,

$\therefore$ it lies completely on the cone.

$\therefore$ (1) is satisfied by the point (tx_1, ty_1, tz_1) for all values of t.

$\therefore$ $t^2(ax_1^2+by_1^2+cz_1^2+2fy_1z_1+2gz_1x_1+2hx_1y_1)$
$+2t(ux_1+vy_1+wz_1)+d=0$ must be an identity.

$\therefore$ $ax_1^2+by_1^2+cz_1^2+2fy_1z_1+2gz_1x_1+2hx_1y_1=0$ ...(2)

$ux_1+vy_1+wz_1=0$ ...(3)

and $d=0$ ...(4)

From (3), we see that if u, v, w are not all zero, the locus of (x_1, y_1, z_1) will be a plane, which is a contradiction, for (x_1, y_1, z_1) is a point on the cone (1).

$\therefore$ $u=v=w=0$.

By (4). $d=0$.

$\therefore$ (1) becomes $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$, which is homogeneous in x, y and z.

This proves the proposition.

7.4. Converse of the theorem given in Art. 7·3.

To prove that every homogeneous equation in x, y, z represents a cone whose vertex is the origin.

Proof. Let the homogeneons equation in x, y, z be

$$f(x, y, z)=0 \qquad \ldots(1)$$

Let $P(x_1, y_1, z_1)$ be **any** point on the locus of (1).

$\therefore$ $f(x_1, y_1, z_1)=0$ $\qquad \ldots(2)$

$\because$ the equation is homogeneous,

$\therefore$ $f(tx_1, ty_1, tz_1)=0$ $\qquad \ldots(3)$

for all values of t.

But (tx_1, ty_1, tz_1) is **any** point on the line OP.

$\therefore$ every point of the line OP lies on the locus of (1),

i.e., OP itself lies on the locus of (1).

$\therefore$ locus of (1) is a cone whose vertex is at the origin. (Art. 7·1)

This proves the proposition.

Aliter. Let the homogeneous equation in x, y, z be

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \qquad \ldots(1)$$

Let $P(x_1, y_1, z_1)$ be **any** point on the locus of (1).

$\therefore$ $ax_1^2+by_1^2+cz_1^2+2fy_1z_1+2gz_1x_1+2hx_1y_1=0.$

Multiplying both sides by t^2, we have

$$a(tx_1)^2+b(ty_1)^2+c(tz_1)^2+2f(ty_1)(tz_1)+(2g(tz_1)(tx_1)$$
$$+2h(tx_1)(ty_1)=0$$

This shows that the point (tx_1, ty_1, tz_1) lies on (1).

But these are the coordinates of **any** point on OP, where **O** is the origin.

$\therefore$ OP itself lies on (1).

$\therefore$ locus of (1) is a cone whose vertex is at the origin.

This proves the proposition.

Aid to memory. The general equation of a cone of second degree in x, y, z whose vertex is at the origin, is

$$\mathbf{ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0.}$$

7·5. If $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ is a generator of the cone represented by the homogeneous equation $f(x, y, z)=0$, to prove that $f(l, m, n)=0$.

Proof. Let $f(x, y, z)=0$ ...(1)

be the equation of cone.

Let $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ be a generator of (1).

Any point on this generator is (tl, tm, tn).

$\because$ the generator lies completely on the cone,

$\therefore$ the point (tl, tm, tn) lies on (1).

$\therefore$ $f(tl, tm, tn)=0$, for all values of t.

But the equation is homogeneous,

$\therefore f(l, m, n)=0$ (Def. Art. 7·2).

This proves the proposition.

Note. *The direction cosines or direction ratios of a generator of a cone satisfy the equation of the cone.*

7·6. Converse of the theorem of Art. 7·5.

To show that, if the direction cosines or direction ratios of a line which passes through a fixed point satisfy a homogeneous equation, the line is a generator of the cone whose vertex is the fixed point.

Proof. Take the fixed point as the origin. Let l, m, n be the direction ratios of the line.

Let these satisfy the homogeneous equation

$$f(l_1, m, n)=0 \qquad ...(1)$$

Now, the equations of the line through the origin with direction ratios l, m, n are

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \qquad ...(2)$$

Eliminating l, m, n between (1) and (2), we get $f(x, y, z)=0$ which, being homogeneous, represents a cone with vertex at the origin.

This proves the proposition.

EXAMPLES VII (A)

Type I. Ex. 1. (i) A plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the coordinate axes in A, B, C. Prove that the equation of the cone generated by lines drawn from O to meet the circle ABC is

$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0.$$

(Punjab, 1957 ; Punjab B.Sc., 1962 ; Delhi Hons., 1962 ; Allahabad, 1962 ; Gujarat, 1956 ; Sagar B.Sc., 1962 ; Vikram, 1962)

(ii) Find the equations of the cone whose vertex is the origin and which passes through the curve given by $ax^2+by^2+cz^2=1$, $\alpha x^2+\beta y^2=2z$.

Sol. (*i*) The equation of the circle ABC is

$$\left.\begin{aligned} &x^2+y^2+z^2-ax-by-cz=0 \qquad &\dots(1)\\ &\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \qquad &\dots(2)\end{aligned}\right\}.$$

Making (1) homogeneous by the help of (2), we have

$$x^2+y^2+z^2-(ax+by+cz)\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)=0,$$

or,
$$x^2+y^2+z^2-x^2-y^2-z^2-xy\left(\frac{a}{b}+\frac{b}{a}\right)-yz\left(\frac{b}{c}+\frac{c}{b}\right)-zx\left(\frac{c}{a}+\frac{a}{c}\right)=0.$$

or,
$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0,$$

which is the required equation of the cone.

(*ii*) Making the two equations of the curve homogeneous in x, y, z, t by introducing proper power of another variable t, we have

$$ax^2+by^2+cz^2=t^2 \qquad \dots(1)$$

and
$$\alpha x^2+\beta y^2=2zt \qquad \dots(2)$$

Eliminating t between (1) and (2), we have

$$ax^2+by^2+cz^2=\left(\frac{\alpha x^2+\beta y^2}{2z}\right)^2,$$

or
$$4z^2(ax^2+by^2+cz^2)=(\alpha x^2+\beta y^2)^2,$$

which is the equation of the required cone.

Ex. 2. Find the equation to the cone whose vertex is the origin and which passes through the curve of intersection of the following :

(*i*) $ax^2+by^2+cz^2=1$, $lx+my+nz=p$.
(Punjab B.Sc., 1958S, 1960 ; Sind, 1948 ; Kashmir, 1957)

(*ii*) $x^2+y^2+z^2+2ax+b=0$, $lx+my+nz=p$.
(Karnatak Engineering, 1961)

(*iii*) $ax^2+by^2=2z$, $lx+my+nz=p$. *(Punjab B.Sc., 1961S)*

[**Ans.** (*i*) $p^2(ax^2+by^2+cz^2)=(lx+my+nz)^2$;
(*ii*) $(x^2+y^2+z^2)p^2+2apx(lx+my+nz)+b(lx+my+nz)^2=0$;
(*iii*) $p(ax^2+by^2)=2z(lx+my+nz)$].

Ex. 3. Show that the equation to the cone whose vertex is the origin and base the curve $f(x, y)=0,\ z=k$ *is* $f\left(\frac{xk}{z}, \frac{yk}{z}\right)=0$.

Ex. 4. Find the equation to the cone whose vertex is the origin and base the circle $x=a, y^2+z^2=b^2$. *(Punjab B.Sc., 1959S)*

Show that the section of the cone by a plane parallel to XOY plane is a hyperbola. *(Delhi Hons., 1954 ; Punjab, 1958S ; Raj. Engi., 1959)*

Type II. Ex. 1. Lines drawn through the point (α, β, γ), whose direction ratios satisfy $al^2+bm^2+cn^2=0$ generate the cone

$$a(x-\alpha)^2+b(y-\beta)^2+c(z-\gamma)^2=0.$$ *(Bombay, 1953 ; Raj. Engi., 1957)*

Sol. Equations of the line through (α, β, γ) is

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \quad \text{.. (1)}$$

The direction cosines atisfy the relation $al^2+bm^2+cn^2=0$ ...(2)

Eliminating l, m, n between (1) and (2), we have

$a(x-\alpha)^2+b(y-\beta)^2+c(z-\gamma)^2=0$, which is the required equation.

Ex. 2. Prove that a cone nf second degree can be found to pass through any five concurrent lines, no three of which are coplanar.

Type III. Ex. 1. Show that the general equation to the cone of the second degree which passes through the axes is $fyz+gzx+hxy=0$.

(Punjab B.Sc., 1961, Raj., 1956 ; Agra 1958 ; Kashmir, 1956 ; Karnatak, 1953 ; Sind (Pakistan), 1948]

Sol. The required cone will have origin as its vertex. The general equation of a cone of the second degree having vertex as origin is

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \quad \text{...(1)}$$

It passes through x-axis,

$\therefore$ direction cosines of x-axis, *viz.*, $(1, 0, 0)$ will satisfy (1).

$\therefore\ a=0$.

Similarly, as the cone passes through the y-axis and z-axis,

we have $b=0, c=0$.

$\therefore$ (1) becomes $fyz+gzx+hxy=0$, which is the required equation.

Ex. 2. Show that a cone of the second degree can be found to pass through any two sets of rectangular axes through the same origin.

(Punjab, 1958 ; Raj., 1956)

Ex. 3. Prove that the equation of the cone through the coordinate axes and the lines in which the plane $lx+my+nz=0$ cuts the cone

$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ is $l(bn^2+cm^2-2fmn)yz$
$+m(cl^2+an^2+2gnl)\,zx+n(am^2+bl^2-2hlm)\,xy=0.$

(Raj., 1961 ; Agra, 1963)

7·7. Cone with given vertex and given conic for base. To find the equation of the cone whose vertex is (α, β, γ) and base the conic $f(x, y) \equiv ax^2+2hxy+by^2+2gx+2fy+c=0,\ z=0$.

The equations of the conic are

$$ax^2+2hxy+by^2+2gx+2fy+c=0,\ z=0 \quad \ldots(1)$$

The equations of any line through (α, β, γ) are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \quad \ldots(2)$$

The line meets $z=0$, where

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=-\frac{\gamma}{n}.$$

$$\therefore\quad x=\alpha-\frac{l}{n}\gamma,\quad y=\beta-\frac{m}{n}\gamma.$$

Substituting these values of x, y in (1), we have

$$a\left(\alpha-\frac{l}{n}\gamma\right)^2+2h\left(\alpha-\frac{l}{n}\gamma\right)\left(\beta-\frac{m}{n}\gamma\right)+b\left(\beta-\frac{m}{n}\gamma\right)^2$$
$$+2g\left(\alpha-\frac{l}{n}\gamma\right)+2f\left(\beta-\frac{m}{n}\gamma\right)+c=0 \quad \ldots(3)$$

Eliminating l, m, n between (2) and (3), the locus of the line (2) is

$$a\left[\alpha-\frac{x-\alpha}{z-\gamma}\gamma\right]^2+2h\left[\alpha-\frac{x-\alpha}{z-\gamma}\gamma\right]\left[\beta-\frac{y-\beta}{z-\gamma}\gamma\right]$$
$$+b\left[\beta-\frac{y-\beta}{z-\gamma}\gamma\right]^2+2g\left[\alpha-\frac{x-\alpha}{z-\gamma}\gamma\right]$$
$$+2f\left[\beta-\frac{y-\beta}{z-\gamma}\gamma\right]+c=0$$

or,

$$a(\alpha z-x\gamma)^2+2h(\alpha z-\gamma x)(\beta z-\gamma y)+b(\beta z-\gamma y)^2$$
$$+2g(\alpha z-\gamma x)(z-\gamma)+2f(\beta z-\gamma y)(z-\gamma)+c(z-\gamma)^2=0,$$

which is the required equation of the cone.

EXAMPLES VII (B)

Ex. 1. Find the equation to the cone whose vertex is (α, β, γ) and base $y^2=4ax,\ z=0$. *(Kashmir; 1954 ; Raj., Engg., 1963 Sup. ; Punjab, 1952 S)*

Sol. Equations of the line through (α, β, γ) is

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \quad \ldots(1)$$

This meets $z=0$, where $x=\alpha-\frac{l}{n}\gamma,\ y=\beta-\frac{m}{n}\gamma.$

Substituting these values in $y^2=4ax$, we have

$$\left(\beta-\frac{m}{n}\gamma\right)^2=4a\left(\alpha-\frac{l}{n}\gamma\right) \qquad ...(2)$$

Eliminating l, m, n between (1) and (2), we have

$$\left(\beta-\frac{y-\beta}{z-\gamma}\gamma\right)^2=4\left(\alpha-\frac{x-\alpha}{z-\gamma}\gamma\right),$$

or $\quad (\beta z-\gamma y)^2=4a(\alpha z-x\gamma)(z-\gamma),$

or $\quad z^2(\beta^2-4a\alpha)-2z\gamma[\beta y-2a(x+\alpha)]+\gamma^2(y^2-4ax)=0,$

which is the required equation of the cone.

Ex. 2. Find the equation of the cone whose vertex is the point (α, β, γ) and whose generating lines pass through the conic

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z=0. \qquad \textit{(Baroda, 1954)}$$

$$\left[\textbf{Ans. } \frac{1}{a^2}(\alpha z-\gamma x)^2+\frac{1}{b^2}(\beta z-\gamma y)^2=(z-\gamma)^2.\right]$$

Ex. 3. The section of a cone whose vertex is P and guiding curve the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $z=0$ by the plane $x=0$ is a rectangular hyperbola. Show that the locus of P is $\frac{x^2}{a^2}+\frac{y^2+z^2}{b^2}=1$. *(Punjab, 1954 S)*

Ex. 4. A cone has as base the circle $z=0$, $x^2+y^2+2ax+2by=0$ and passes through the fixed point $(0, 0, c)$ on the z-axis. If the section of the cone by the plane ZOX is a rectangular hyperbola, prove that the vertex lies on a fixed circle. *(Banaras, 1954 ; North Bengal Hons., 1964)*

Ex. 5. Two cones pass through the curves $y=0$, $z^2=4ax$; $x=0$, $z^2=4by$, and they have a common vertex; the plane $z=0$ meets them in two conics that intersect in four concyclic points. Show that the vertex lies on the surface

$$z^2\left(\frac{x}{a}+\frac{y}{b}\right)=4(x^2+y^2). \qquad \textit{(Karnatak, 1961 ; Agra, 1957)}$$

Ex. 6. Find the equation of the cone whose vertex is the point (1, 1, 0) and whose guiding curve is $y=0$, $x^2+z^2=4$. *(Jodhpur Engg., 1965 Sup.)*

[**Ans.** $x^2-3y^2+z^2-2xy+8y-4=0$.]

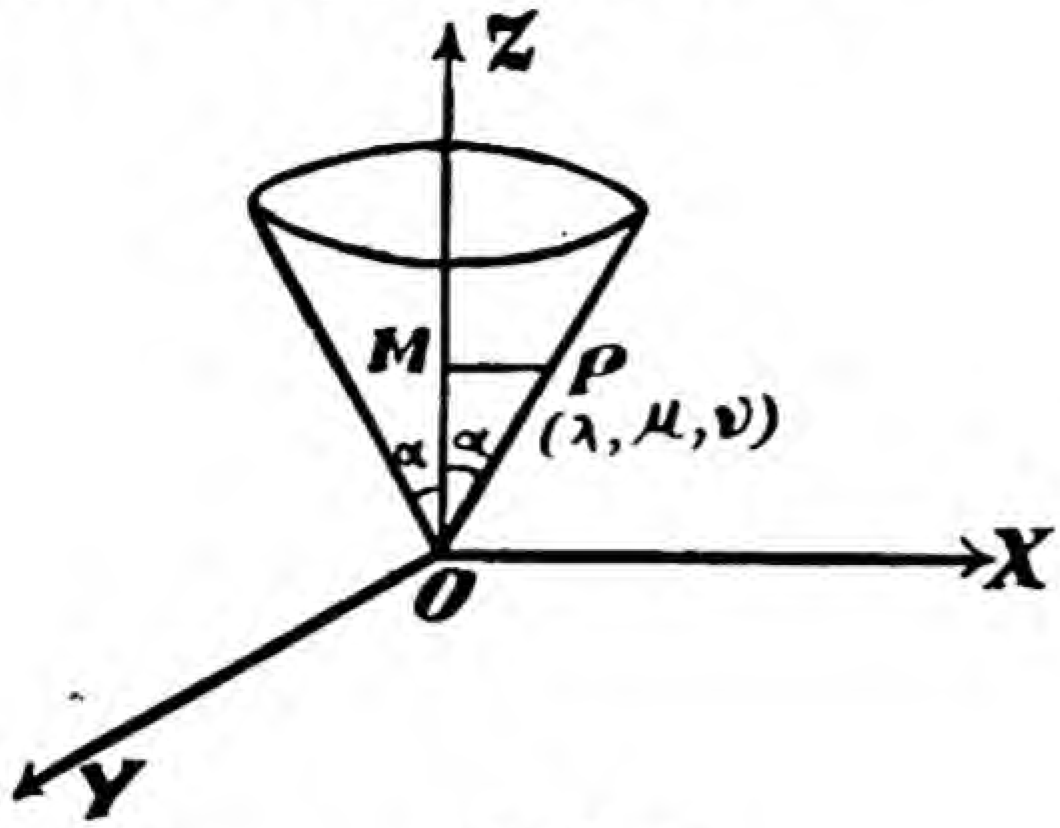
7·8. Right circular cone : Def.

A right circular cone is a surface generated by a straight line which passes through a fixed point (called the **vertex** of the cone), and makes a constant angle (called the **semi-vertical** angle of the cone) with a fixed line (called the **axis** of the cone) through that fixed point.

7·9. Standard equation.

To find the equation of the right circular cone whose vertex is the origin, axis the z-axis and semi-vertical angle α.

Let P (λ, μ, ν) be **any** point on the cone.



It is given that $\angle POZ = \alpha$.

Draw PM pependicular to OZ.

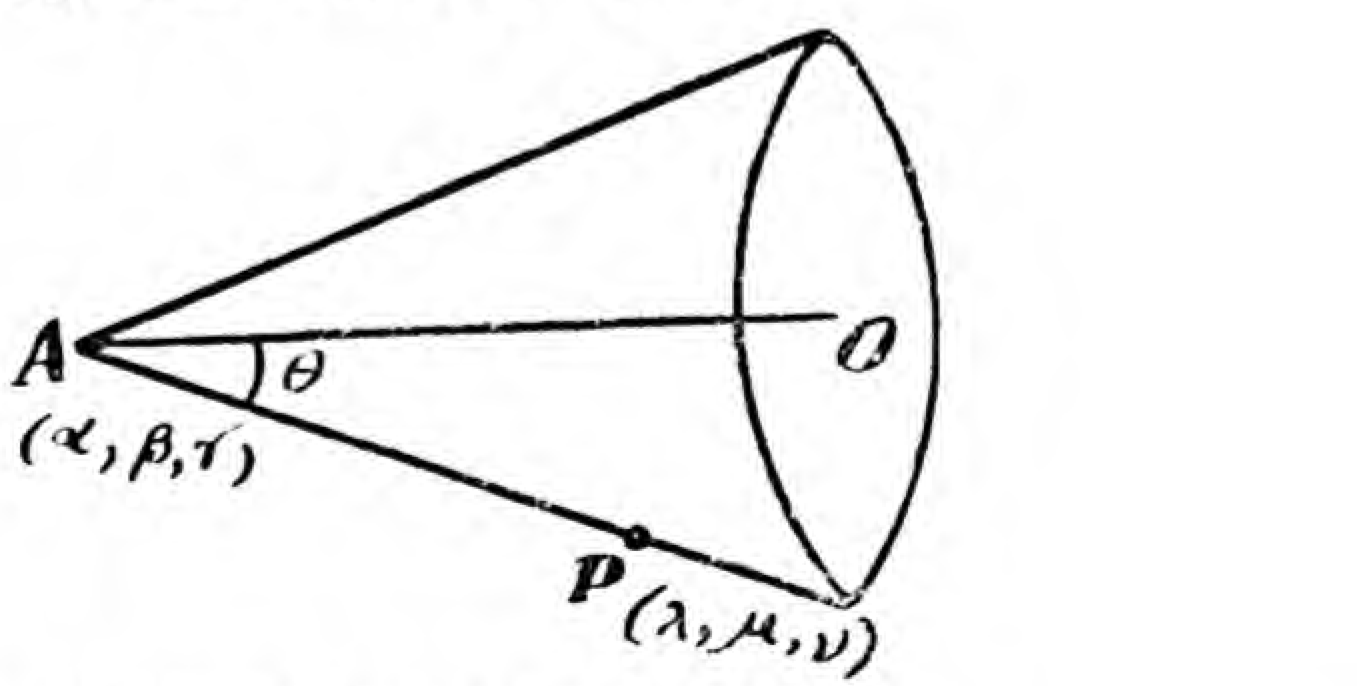
$$\therefore \qquad \tan \alpha = \frac{PM}{OM} = \frac{\sqrt{\lambda^2 + \mu^2}^*}{\nu}$$

or $$\lambda^2 + \mu^2 = \nu^2 \tan^2 \alpha.$$

$\therefore$ locus of (λ, μ, ν) is $\mathbf{x^2 + y^2 = z^2 \tan^2 \alpha}$, which is the required equation of the cone.

7.10. To find the equation of the right circular cone with its vertex at (α, β, γ), its axis the line $\dfrac{x-\alpha}{l} = \dfrac{y-\beta}{m} = \dfrac{z-\gamma}{n}$ and its semi-vertical angle θ, ; l, m, n being the direction ratios.

Let A (α, β, γ) be the vertex of the cone.



Let $P(\lambda, \mu, \nu)$ be **any** point on the cone.

---

* $PM^2 = OP^2 - OM^2 = (\lambda^2 + \mu^2 + \nu^2) - (\text{projection of OP on OZ})^2$
$= (\lambda^2 + \mu^2 + \nu^2) - (\nu)^2 = \lambda^2 + \mu^2$

and $OM = \text{projection of OP on } OZ = \lambda.0 + \mu.0 + \nu.1 = \nu.$

Let AP make an angle θ with the axis AO of the cone.

Direction ratios of AP are $\lambda-\alpha, \mu-\beta, \nu-\gamma$.

Direction ratios of AO are l, m, n.

$$\therefore \quad \cos\theta=\frac{l(\lambda-\alpha)+m(\mu-\beta)+n(\nu-\gamma)}{\sqrt{l^2+m^2+n^2}\sqrt{(\lambda-\alpha)^2+(\mu-\beta)^2+(\nu-\gamma)^2}}$$

$\therefore$ locus of (λ, μ, ν) is

$$(l^2+m^2+n^2)[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2]\cos^2\theta$$
$$=[l(x-\alpha)+m(y-\beta)+n(z-\gamma)]^2,$$

which is the required equation.

EXAMPLES VII (C)

Type I. Ex. 1. Find the direction cosines of the axis of the right circular cone which passes through the lines drawn from O with direction cosines proportional to (3, 6, −2), (2, 2, −1), (−1, 2, 2) and prove that the cone also passes through the coordinate axes. (*Vikram Engg., 1960*)

Sol. Let l, m, n be the direction cosines of the axis of the cone.

Let P, Q and R be the points $(3, 6, -2)$, $(2, 2, -1)$ and $(-1, 2, 2)$ respectively.

$\therefore$ direction cosines of OP are $\frac{3}{7}, \frac{6}{7}, \frac{-2}{7}$, direction cosines of OQ are $\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}$, and direction cosines of OR are $-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}$.

$\because$ the axis of the cone makes the same angle with each generator,

$$\therefore \quad \frac{3l}{7}+\frac{6m}{7}-\frac{2n}{7}=\frac{2l}{3}+\frac{2m}{3}-\frac{n}{3}$$
$$=-\frac{l}{3}+\frac{2m}{3}+\frac{2n}{3}=\cos\alpha,$$

say, where α is the semi-vertical angle of the cone.

From the last two relations, we have $l=n$.

From first two relations, we have

$$\frac{l}{7}+\frac{6m}{7}=\frac{l}{3}+\frac{2m}{3},$$

or, $\quad 3l+18m=7l+14m, \quad$ or, $\quad 4l=4m. \quad$ or, $\quad l=m.$

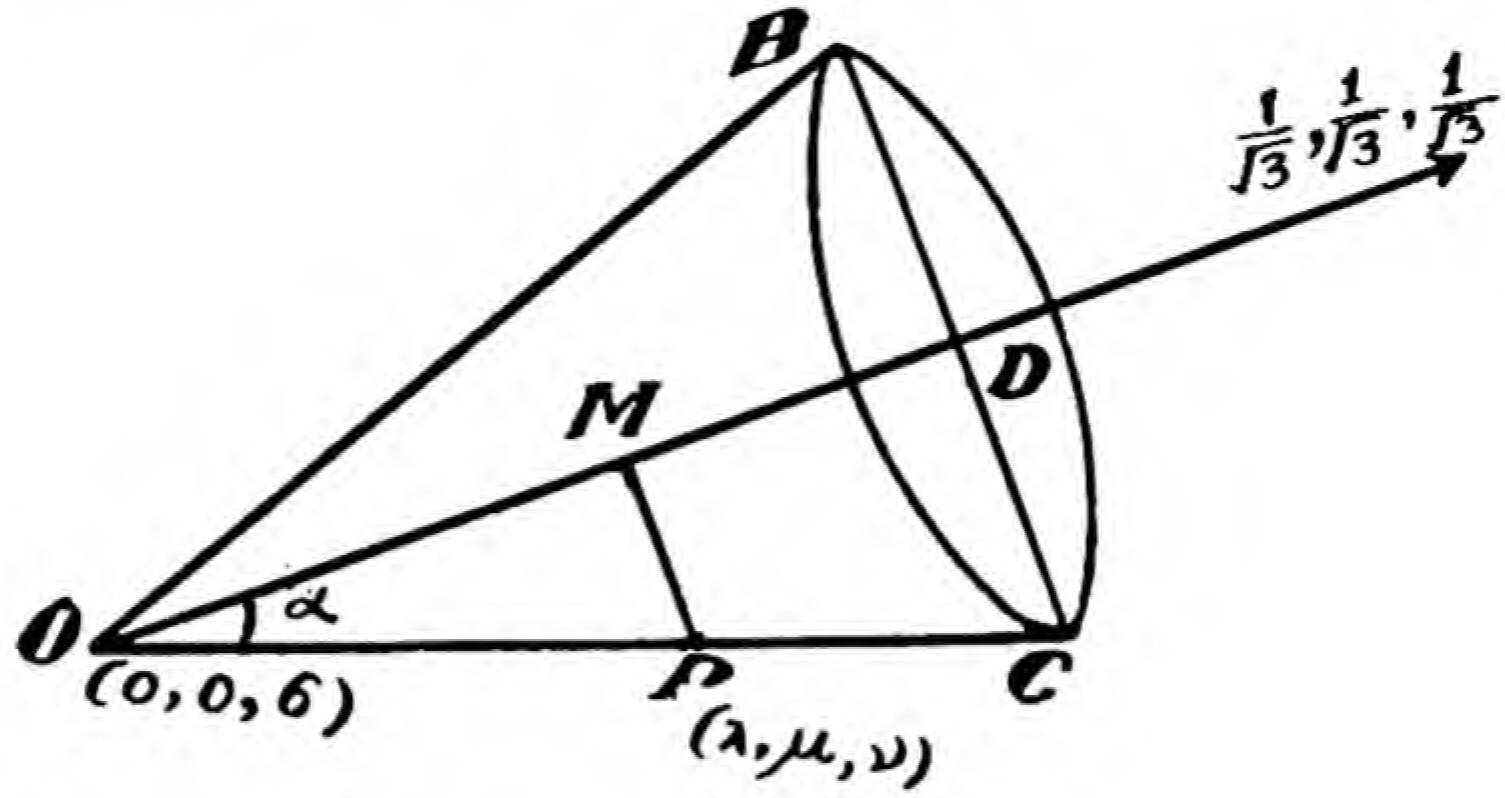
$$\therefore \quad l=m=n=\frac{1}{\sqrt{3}}.$$

$\therefore$ direction cosines of the axis are

$$\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}.$$

Now, $\cos\alpha=\frac{1}{\sqrt{3}}\left(\frac{3}{7}+\frac{6}{7}-\frac{2}{7}\right)=\frac{1}{\sqrt{3}}$

Let P (λ, μ, ν) be any point on the cone.



Draw PM perpendicular to the axis OD.

$$\therefore \qquad \sin \alpha = \frac{PM}{OP}, \qquad \text{or,} \qquad PM^2 = OP^2 \sin^2 \alpha,$$

$$\text{or,} \qquad PM^2 = (\lambda^2+\mu^2+\nu^2).\tfrac{2}{3} \left(\because \cos \alpha = \frac{1}{\sqrt{3}} \right) \qquad \ldots(1)$$

Now, PM^2 = Square of the distance of $P(\lambda, \mu, \nu)$ from the line OD having direction cosines $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

$$= [(\mu-\nu)^2+(\nu-\lambda)^2+(\lambda-\mu)^2]\tfrac{1}{3}.$$

$\therefore$ (1) becomes

$$\tfrac{1}{3}[(\mu-\nu)^2+(\nu-\lambda)^2+(\lambda-\mu)^2] = \tfrac{2}{3}(\lambda^2+\mu^2+\nu^2).$$

$\therefore$ locus of (λ, μ, ν) is

$$(y-z)^2+(z-x)^2+(x-y)^2 = 2(x^2+y^2+z^2).$$

$$\text{or,} \qquad y^2+z^2-2yz+z^2+x^2-2xz+x^2+y^2-2xy = 2x^2+2y^2+2z^2,$$

$$\text{or,} \qquad xy+yz+zx=0. \qquad \ldots(2)$$

This is the equation of the cone.

The direction cosines of the axes satisfy (2).

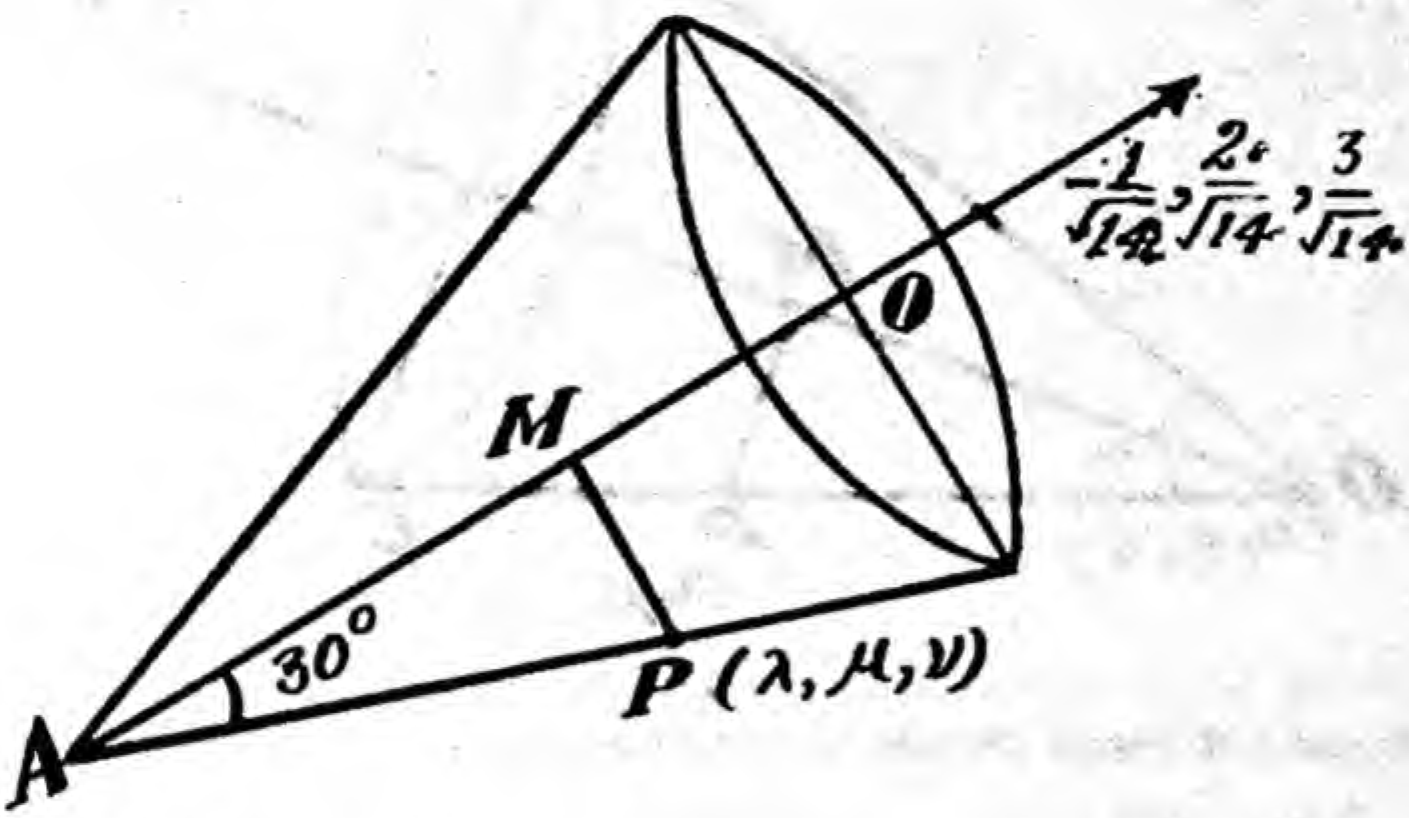
$\therefore$ it passes through the axes.

Ex. 2. Lines are drawn from O with direction—cosines proportional to (1, 2, 2) (2, 3, 6), (3, 4, 12). Prove that the axis of the right circular cone through them has direction cosines $-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$, and that the semi-vertical angle of the cone is $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$. (*Nagpur, T. D. C., 1962 ; Gujarat Engg., 1964*)

Type II. Ex. 1. Find the equation of a right circular cone whose vertex is the point (1, 1, 1), the axis is the line given by the equation $\frac{x-1}{-1} = \frac{y-1}{2} = \frac{z-1}{3}$ **and semi-vertical angle is 30°.** (*Raj. Engg., 1962*)

Sol. The direction cosines of the axis are

$$-\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}.$$



Let P(λ, μ, ν) be **any** point on the cone and A be the vertex.

Direction ratios of AP are $(\lambda-1), (\mu-1), (\nu-1)$.

$$\therefore \cos 30^\circ = \frac{-(\lambda-1)+2(\mu-1)+3(\nu-1)}{\sqrt{14}\ \sqrt{(\lambda-1)^2+(\mu-1)^2+(\nu-1)^2}}$$

$\therefore$ locus of (λ, μ, ν) is

$$14\cdot\tfrac{3}{4}\,[(x-1)^2+(y-1)^2+(z-1)^2]=(-x+2y+3z-4)^2$$

or, $$21\,[x^2+y^2+z^2-2x-2y-2z+3]=2[(-x+2y)^2+(3z-4)^2+2(-x+2y)(3z-4)]$$

or $$21x^2+21y^2+21z^2-42x-42y+63$$
$$=2x^2+8y^2-8xy+18z^2+32-48z-12xz+24yz+16x-32y,$$

or, $$19x^2+13y^2+3z^2+8xy+12xz-24yz-58x-10y+6z+31=0,$$

which is the required equation of the cone.

Ex. 2. Find the equation of the right circular cone whose vertex is at the origin, whose axis is the line $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$, and which has a vertical angle of 60°. *(Gujarat, 1950)*

[**Ans.** $19x^2+13y^2+3z^2-24yz-12zx-8xy=0$.]

Ex. 3. Find the equation to the right circular cone whose vertex is (2, −3, 5), axis line, PQ, through P making equal angles with the axes, and semi-vertical angle is 30°. *(Bombay, 1952 ; Raj. Engg., 1951)*

[**Ans.** $4\{(y-z+8)^2+(z-x-3)^2+(x-y-5)^2\}=3\{(x-2)^2+(y+3)^2+(z-5)^2\}$]

Ex. 4. The axis of a right cone, vertex origin O, makes equal angles with the coordinate axes, and the cone passes through the line drawn from O with direction cosines proportional to (1, −2, 2). Find the equation to the cone.

(Baroda Engg., 1961 ; Punjab, 1960S)

[**Ans.** $4x^2+4y^2+4z^2+9yz+9zx+9xy=0$.]

7·11. Enveloping cone of the sphere : Def.

The locus of the tangent lines to a sphere drawn from a given point is a cone called the enveloping cone of the sphere having the given point as its vertex.

7·12. To find the equation of the enveloping cone of the sphere $x^2+y^2+z^2=a^2$ with its vertex at (x_1, y_1, z_1)

The equation of the sphere is $x^2+y^2+z^2=a^2$ ...(1)

Let the given point be $A(x_1, y_1, z_2)$.

Let P (x, y, z) be **any** point on a tangent drawn from A to the sphere.

The coordinates of a point dividing AP in the ratio $m : 1$ is

$$\left(\frac{mx+x_1}{m+1}, \frac{my+y_1}{m+1}, \frac{mz+z_1}{m+1}\right).$$

If it lies on (1), then

$$\left(\frac{mx+x_1}{m+1}\right)^2+\left(\frac{my+y_1}{m+1}\right)^2+\left(\frac{mz+z_1}{m+1}\right)^2=a^2,$$

or $$m^2(x^2+y^2+z^2-a^2)+2m(xx_1+yy_1+zz_1-a^2)+(x_1^2+y_1^2+z_1^2-a^2)=0 \quad ...(2)$$

$\because$ AP touches (1),

$\therefore$ (2) has equal roots, condition for which is

$$4(xx_1+yy_1+zz_1-a^2)^2=4(x^2+y^2+z^2-a^2)(x_1^2+y_1^2+z_1^2-a^2)$$

or, $$(\mathbf{x^2+y^2+z^2-a^2})(\mathbf{x_1^2+y_1^2+z_1^2-a^2})=(\mathbf{xx_1+yy_1+zz_1-a^2})^2,$$

which is the required equation.

Note 1. A useful notation.

Let S denote the L.H.S. of the equation of the given sphere whose R.H.S. is zero, S_1 denote the expression S after substituting the coordinates of the given point in it, and T denote the L.H.S. of the tangent plane at the given point, whose R.H.S. is zero.

$\therefore$ *the equation of the enveloping cone of the sphere $x^2+y^2+z^2=a^2$ whose vertex is (x_1, y_1, z_1) is*

$$\mathbf{SS_1=T^2}, \text{ where } \mathbf{S\equiv x^2+y^2+z^2-a^2} \quad \mathbf{S_1\equiv x_1^2+y_1^2+z_1^2-a^2}.$$
$$\mathbf{T\equiv[xx_1+yy_1+zz_1-a^2]}.$$

Note 2. *A similar result is true for general sphere also.*

EXAMPLES VII (D)

Ex. 1. Find enveloping cone of the sphere $x^2+y^2+z^2+2x-2y=2$ with its vertex at (1, 1, 1).

Sol. Here $S\equiv x^2+y^2+z^2+2x-2y-2$, $S_1\equiv 1+1+1+2-2-2=1$

$$T=x(1)+y(1)+z(1)+1(x+1)-1(y+1)-2.$$
$$=x+y+z+x+1-y-1-2=2x+z-2.$$

$\therefore$ equation of the enveloping cone is

$$(x^2+y^2+z^2+2x-2y-2)(1)=(2x+z-2)^2,$$

or, $$x^2+y^2+z^2+2x-2y-2=4x^2+z^2+4+4zx-4z-8x,$$

or, $$3x^2-y^2+4zx-10x+2y-4z+6=0.$$

Ex. 2. Prove that the lines drawn from the origin so as to touch the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ lie on the cone $d(x^2+y^2+z^2)$ $=(ux+vy+wz)^2$ *(Punjab, 1957S)*.

Ex. 3. Show that the plane $z=0$ cuts the enveloping cone of the sphere $x^2+y^2+z^2=1$ which has its vertex at (2, 4, 1) in a rectangular hyperbola.

7·13. To find the condition that the general equation of the second degree may represent a cone.

Let $ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$ ...(1)
represent a cone with vertex at (x_1, y_1, z_1).

Shifting the origin to (x_1, y_1, z_1), we have the transformed equation as

$$a(x+x_1)^2+b(y+y_1)^2+c(z+z_1)^2+2f(y+y_1)(z+z_1)$$
$$+2g(z+z_1)(x+x_1)+2h(x+x_1)(y+y_1)+2u(x+x_1)$$
$$+2v(y+y_1)+2w(z+z_1)+d=0,$$

or, $$ax^2+by^2+cz^2+2fyz+2gzx+2hxy$$
$$+2[x(ax_1+hy_1+gz_1+u)+y(hx_1+by_1+fz_1+v)$$
$$+z(gx_1+fy_1+cz_1+w)]+(ax_1^2+by_1^2+cz_1^2+2fy_1z_1$$
$$+2gz_1x_1+2hx_1y_1+2ux_1+2vy_1$$
$$+2wz_1+d)=0 \quad ...(2)$$

$\because$ This equation is referred to vertex as the origin,

$\therefore$ it must be homogeneous equation.

$\therefore$ coefficient of x in (2)=0, coefficient of y in (2)=0, coefficient of z in (2)=0, absolute term in (2)=0.

or $$ax_1+hy_1+gz_1+u=0 \quad ...(3)$$
$$hx_1+by_1+fz_1+v=0 \quad ...(4)$$
$$gx_1+fy_1+cz_1+w=0 \quad ...(5)$$

and $$ax_1^2+by_1^2+cz_1^2+2fy_1z_1+2gz_1x_1+2hx_1y_1+2ux_1+2vy_1$$
$$+2wz_1+d=0 \quad ...(6)$$

From (6), we have

$$x_1(ax_1+hy_1+gz_1+u)+y_1(hx_1+by_1+fz_1+v)$$
$$+z_1(gx_1+fy_1+cz_1+w)+(ux_1+vy_1+wz_1+d)=0$$

or,
$$ux_1+vy_1+wz_1+d=0 \qquad \ldots(7),$$

using (3), (4) and (5).

Eliminating x_1, y_1, z_1 between (3), (4), (5) and (7), we have

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix}=0,$$ which is the required condition.

Note 1. To find the vertex of the cone.

From equations (3), (4) and (5), we have

$$\frac{x_1}{\begin{vmatrix} h & g & u \\ b & f & v \\ f & c & w \end{vmatrix}}=\frac{-y_1}{\begin{vmatrix} a & g & u \\ h & f & v \\ g & c & w \end{vmatrix}}$$

$$=\frac{z_1}{\begin{vmatrix} a & h & u \\ h & b & v \\ g & f & w \end{vmatrix}}=\frac{-1}{\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}},$$

which gives the vertex (x_1, y_1, z_1), *provided*

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0.$$

Note 2. If $F(x, y, z) \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$ represents a cone, the coordinates of its vertex satisfy the equations $F_x=0$, $F_y=0$, $F_z=0$, $F_t=0$, where t is used to make F(x, y, z) homogeneous and is equated to unity after differenciation,

$$\left(F_x=\frac{\partial F}{\partial x},\ F_y=\frac{\partial F}{\partial y},\ F_z=\frac{\partial F}{\partial z},\ F_t=\frac{\partial F}{\partial t}\right).$$

Making $F(x, y, z)$ homogeneous in x, y, z, t by introducing proper powers of t, we have

$$F(x, y, z, t) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2uxt + 2vyt + 2wzt + dt^2$$

Now $F_x = 2(ax + gz + hy + ut)$, $F_y = 2(by + fz + hx + vt)$,
$F_z = 2(cz + fy + gx + wt)$, $F_t = 2(ux + vy + wz + dt)$.

Putting $t=1$, and equating to zero, we have

$$ax + hy + gz + u = 0,\ hx + by + fz + v = 0,$$
$$gx + fy + cz + w = 0,\ ux + vy + wz + d = 0.$$

$\therefore$ *we see from (3), (4), (5) and (7) that the vertex (x_1, y_1, z_1) satisfies the equations*

$$F_x = 0,\ F_y = 0,\ F_z = 0,\ F_t = 0,$$

where t is put equal to unity after differentiation.

EXAMPLES VII (E)

Ex. 1. Prove that the equation $2y^2 - 8yz - 4zx - 8xy + 6x - 4y - 2z + 5 = 0$ represents a cone whose vertex is

$$\left(-\frac{7}{6}, \frac{1}{3}, \frac{5}{6}\right)$$ (*Delhi Hons., 1950*)

Sol. Here $a=0$, $b=2$, $c=0$, $f=-4$, $g=-2$, $h=-4$, $u=3$, $v=-2$, $w=-1$, $d=5$.

$$\therefore \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} = \begin{vmatrix} 0 & -4 & -2 & 3 \\ -4 & 2 & -4 & -2 \\ -2 & -4 & 0 & -1 \\ 3 & -2 & -1 & 5 \end{vmatrix}$$

$$=4\begin{vmatrix} -4 & -4 & -2 \\ -2 & 0 & -1 \\ 3 & -1 & 5 \end{vmatrix} -2\begin{vmatrix} -4 & 2 & -2 \\ -2 & -4 & -1 \\ 3 & -2 & 5 \end{vmatrix} -3\begin{vmatrix} -4 & 2 & -4 \\ -2 & -4 & 0 \\ 3 & -2 & -1 \end{vmatrix}$$

$$=4[-4(-1)+4(-10+3)-2(2)]-2[-4(-20-2)-2(-10+3)-2(4+12)]$$
$$-3[-4(4)-2(2)-4(4+12)]$$
$$=[4-28-4]-2[88+14-32]-3[-16-4-64]=0.$$

$\therefore$ the given equation represents a cone.

Let $\quad F(x, y, z) \equiv 2y^2 - 8yz + 4zx - 8xy + 6x - 4y - 2z + 5 = 0.$

Making F homogeneous by introducing suitable powers of t, we have

$F(x, y, z, t) = 2y^2 - 8yz - 4zx - 8xy + 6xt - 4yt - 2zt + 5t^2$

$$\therefore \quad F_x = -4z - 8y + 6t \qquad \therefore \quad (F_x)_{t=1} = -4z - 8y + 6$$

$$F_y = 4y - 8z - 8x - 4t \qquad \therefore \quad (F_y)_{t=1} = 4y - 8z - 8x - 4$$

$$F_z = -8y - 4x - 2t \qquad \therefore \quad (F_z)_{t=1} = -8y - 4x - 2$$

$$F_t = 6x - 4y - 2z + 10t \qquad \therefore \quad (F_t)_{t=1} = 6x - 4y - 2z + 10.$$

$\therefore$ vertex is given by

$$4z + 8y - 6 = 0, \quad 4y - 8z - 8x - 4 = 0$$

and

$$-8y - 4x - 2 = 0.$$

$\therefore$ vertex is $\quad \left(-\frac{7}{6}, \frac{1}{3}, \frac{5}{6}\right).$

This satisfies $(F_t)_{t=1} = 0.$

Ex. 2. Prove that the equation

$$7x^2 + 2y^2 + 2z^2 - 10zx + 10xy + 26x - 2y + 2z - 17 = 0$$

represents a cone whose vertex is $(1, -2, 2)$. *(Punjab B.Sc., 1961)*

Ex. 3. Prove that the equation

$$2x^2 + 2y^2 + 7z^2 - 10yz - 10zx + 2x + 2y + 26z - 17 = 0$$

represents a cone with vertex at $(2, 2, 1)$. *(Karnatak, 1959)*

SECTION II

A LINE AND A CONE. TANGENT PLANE. CONDITION OF TANGENCY

7·14. To find the points of intersection of a line and a cone.

Let the line be

$$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n} \qquad \ldots(1)$$

and the cone be

$$f(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \qquad \ldots(2)$$

Any point on (1) is

$$(\alpha + lr, \beta + mr, \gamma + nr).$$

It will lie on (2) if

$$a(\alpha+lr)^2+b(\beta+mr)^2+c(\gamma+nr)^2+2f(\beta+mr)(\gamma+nr)$$
$$+2g(\gamma+nr)(\alpha+lr)+2h(\alpha+lr)(\beta+mr)=0.$$

or, $$r^2(al^2+bm^2+cn^2+2fmn+2gnl+2hlm)$$
$$+2r[l(a\alpha+h\beta+g\gamma)+m(h\alpha+b\beta+f\gamma)$$
$$+n(g\alpha+f\beta+c\gamma)]+f(\alpha, \beta, \gamma)=0.$$

This is a quadratic equation in r corresponding to each of which we have a point common to (1) and (2).

$\therefore$ every line meets a cone in two points.

7·15. To find the tangent line and tangent plane at the point (α, β, γ) of the cone $\mathbf{ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0}$.

Let $$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad \ldots(1)$$

be **any** line through the point (α, β, γ) of the cone

$$f(x, y, z)\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \qquad \ldots(2)$$

Any point on (1) is $(\alpha+lr, \beta+mr, \gamma+nr)$.

It lies on (2) if

$$r^2[f(l, m, n)]+2r[l(a\alpha+h\beta+g\gamma)+m(h\alpha+b\beta+f\gamma)$$
$$+n(g\alpha+f\beta+c\gamma)]+f(\alpha, \beta, \gamma)=0 \qquad \ldots(3)$$

This is a quadratic equation in r.

$\because$ (α, β, γ) lies on (1),

$\therefore$ $f(\alpha, \beta, \gamma)=0$.

$\therefore$ one root of (3) is zero.

If the line (1) touches (2), then the other root of (3) is also zero, *i.e.*, coefficient of $r=0$,

i.e., $$l(a\alpha+h\beta+g\gamma)+m(h\alpha+b\beta+f\gamma)+n(g\alpha+f\beta+c\gamma)=0 \qquad \ldots(4)$$

Hence the line (1) corresponding to the set of values of l, m, n satisfying (4) is a **tangent line at (α, β, γ) of the cone (2).**

Eliminating l, m, n between (1) and (4), we get the locus of all the tangent lines through (α, β, γ) *i.e.*, the equation of the tangent plane at (α, β, γ), *viz.*, $(x-\alpha)(a\alpha+h\beta+g\gamma)+(y-\beta)(h\alpha+b\beta+f\gamma)$
$+(z-\gamma)(g\alpha+f\beta+c\gamma)=0$,

or, $$\mathbf{x(a\alpha+h\beta+g\gamma)+y(h\alpha+b\beta+f\gamma)+z(g\alpha+f\beta+c\gamma)=0.}$$

Cor. 1. The tangent plane at any point of the cone passes through its vertex.

Cor. 2. The tangent plane at any point of a cone touches it at all points of the generator through that point, *i.e.*, touches along a ganerator.*

7·16. A useful notation.

Let (1) $\mathbf{D}=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=abc+2fgh-af^2-bg^2-ch^2.$

(2) **A, B, C, F, G, H** are the cofactors of a, b, c, f, g, h respectively in the determinant **D**.

$$\therefore \qquad \mathbf{A}=bc-f^2,\ \mathbf{B}=ca-g^2,\ \mathbf{C}=ab-h^2\ ;$$
$$\mathbf{F}=gh-af,\ \mathbf{G}=hf-bg,\ \mathbf{H}=fg-ch,$$

7·17. Condition of tangency.

To find the condition that the plane $\mathbf{lx+my+nz=o}$ may touch the cone $\mathbf{ax^2+by^2+cz^2+2fyz+2gzx+2hxy=o}$.

Let the plane $\qquad lx+my+nz=0 \qquad$ ...(1)

touch the cone $\qquad ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \qquad$ ...(2)

at the point (α, β, γ).

The equation of the tangent plane at (α, β, γ) of the cone (2) is
$x(a\alpha+h\beta+g\gamma)+y(h\alpha+b\beta+f\gamma)+z(g\alpha+f\beta+c\gamma)=0 \qquad$ ...(3)

Now (3) is the same as (1).

$\therefore$ comparing coefficients of like terms, we have

$$\frac{a\alpha+h\beta+g\gamma}{l}=\frac{h\alpha+b\beta+f\gamma}{m}=\frac{g\alpha+f\beta+c\gamma}{n}=-\lambda, \text{ say.}$$

$$\therefore \qquad a\alpha+h\beta+g\gamma+l\lambda=0 \qquad \text{...(4)}$$

*Equation of the generator OP, through $P(\alpha, \beta, \gamma)$ is $\dfrac{x}{\alpha}=\dfrac{y}{\beta}=\dfrac{z}{\gamma}=r$ say.

Any point on this generator is $(\alpha r, \beta r, \gamma r)$. The tangent plane at this point is

$$\alpha r\frac{\partial F}{\partial x}+\beta r\frac{\partial F}{\partial y}+\gamma r\frac{\partial F}{\partial z}=0$$

or

$$\alpha\frac{\partial F}{\partial x}+\beta\frac{\partial F}{\partial y}+\gamma\frac{\partial F}{\partial z}=0,$$

which is the same as the tangent plane at (α, β, γ), $\therefore$ etc.

$$h\alpha+b\beta+f\gamma+\lambda m=0 \quad ...(5)$$

and $$g\alpha+f\beta+c\gamma+n\lambda=0 \quad ...(6)$$

Also, $\because$ (α, β, γ) lies on (1),

$\therefore$ $$l\alpha+m\beta+n\gamma=0 \quad ...(7)$$

Eliminating $\alpha, \beta, \gamma, \lambda$ between (4), (5), (6) and (7), we have

$$\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & o \end{vmatrix}=0, \text{ which is the required condition.}$$

or, using the notation of Art. 7·16, we have

$$-(\mathbf{A}l^2+\mathbf{B}m^2+\mathbf{C}n^2+2\mathbf{F}mn+2\mathbf{G}nl+2\mathbf{H}lm)=0,$$

or, $$\mathbf{A}l^2+\mathbf{B}m^2+\mathbf{C}n^2+2\mathbf{F}mn+2\mathbf{G}nl+2\mathbf{H}lm=0.$$

EXAMPLES VII (F)

Ex. 1. Find the points in which the line

$$\frac{x+1}{-1}=\frac{y-12}{5}=\frac{z-7}{2}$$

cuts the surface $11x^2-5y^2+z^2=0$. [*Punjab (Pakistan), 1957*]

Sol. Any point on the given line is

$$(-1-r, \quad 12+5r, \quad 7+2r).$$

It lies on the cone $11x^2-5y^2+z^2=0$.

$\therefore$ $11(-1-r)^2-5(12+5r)^2+(7+2r)^2=0,$

or, $r^2+5r+6=0 \quad \therefore \ r=-2, -3,$

$\therefore$ points of intersection are (1, 2, 3) ; (2, −3, 1).

Ex. 2. P, Q are the points of intersection of the line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$

with the cone $ax^2+by^2+cz^2=0$. Show that the sphere described on PQ as diameter will pass through the vertex of the cone, if

$$a(\mu^2+\nu^2)+b(\nu^2+\lambda^2)+c(\lambda^2+\mu^2)=0,$$

where $\lambda=\beta n-\gamma l, \ \mu=\gamma l-\alpha n, \ \nu=\alpha m-\beta l.$

SECTION III

CONE AND A PLANE THROUGH ITS VERTEX

7·18. A notation.

We shall use $\mathbf{P}^2$ to denote

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & o \end{vmatrix},$$

or, $$-(\mathbf{A}u^2+\mathbf{B}v^2+\mathbf{C}w^2+2\mathbf{F}vw+2\mathbf{G}wu+2\mathbf{H}uv).$$

7·19. To find the angle between the lines in which the plane $\mathbf{ux+vy+wz=o}$ cuts the cone

$$\mathbf{f(x, y, z)\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=o.}$$

Let the cone be $f(x, y, z)$

$$\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \qquad ...(1)$$

and the plane be $ux+vy+wz=0$ ...(2)

Let $$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \qquad ...(3)$$

be the equations of a line of section.

$\because$ (3) lies in (2),

$\therefore$ it is perpendicular to the normal to the plane (2).

$\therefore$ $ul+vm+wn=0.$ ...(4)

Also, (3) lies on (1),

$\therefore$ it is a generator of the cone, *i.e.*, its direction cosines satisfy the equation of the cone.

$\therefore$ $al^2+bm^2+cn^2+2fmn+2gnl+2hlm=0.$ ...(5)

Eliminating n between (4) and (5), we have

$$al^2+bm^2+c\left(\frac{ul+vm}{w}\right)^2+2fm\left(-\frac{ul+vm}{w}\right)$$
$$+2gl\left(-\frac{ul+vm}{w}\right)+2hlm=0.$$

or, $$al^2w^2+bm^2w^2+c(u^2l^2+v^2m^2+2uvlm)-2fmw(ul+vm)$$
$$-2glw(ul+vm)+2hlmw^2=0,$$

or, $$l^2(aw^2+cu^2-2guw)+2lm(cuv-fuw-gvw+hw^2)+m^2(bw^2$$
$$+cv^2-2fvw)=0.$$

or, $$\frac{l^2}{m^2}(aw^2+cu^2-2guw)+2\frac{l}{m}(cuv-fvw-gvw+hw^2)$$
$$+(bw^2+cv^2-2fvw)=0. \qquad ...(6)$$

Now, (6) is a quadratic equation in $\frac{l}{m}$. This shows that (2) cuts (1) in two straight lines.

If l_1, m_1, n_1 ; l_2, m_2, n_2 be the direction cosines of the lines of section of (1) and (2), then $\frac{l_1}{m_1}$ and $\frac{l_2}{m_2}$ are the roots of (6).

$$\therefore \quad \frac{l_1l_2}{m_1m_2}=\frac{bw^2+cv^2-2fvw}{aw^2+cu^2-2guw} \text{ and } \frac{l_1}{m_1}+\frac{l_2}{m_2}$$
$$=-\frac{2(cuv-fuw-gvw+hw^2)}{aw^2+cu^2-2guw}$$

or, $$\frac{l_1l_2}{bw^2+cv^2-2fvw}=\frac{m_1m_2}{aw^2+cu^2-2guw}$$

and $$\frac{l_1m_2+l_2m_1}{-2(cuv-fuw-gvw+hw^2)}=\frac{m_1m_2}{aw^2+cu^2-2guw}$$

$$\therefore \quad \frac{l_1l_2}{bw^2+cv^2-2fvw}=\frac{m_1m_2}{aw^2+cu^2-2guw}$$
$$=\frac{l_1m_2+l_2m_1}{-2(cuv-fuw-gvw+hw^2)}$$
$$=\frac{\sqrt{(l_1m_2+l_2m_1)^2-4l_1l_2m_1m_2}}{\sqrt{4(cuv-fuw-gvw+hw^2)^2-4(bw^2+cv^2-2fvw)(aw^2+cu^2-2guw)}}$$
$$...(7)$$

Now, $$4(cuv-fuw-gvw+hw^2)^2$$
$$-4(bw^2-cv^2-2fvw)(cu^2+aw^2-2gwu)$$
$$=4[c^2u^2v^2+f^2w^2u^2+g^2v^2w^2+h^2w^4-2cfu^2vw$$
$$-2cguv^2w+2chuvw^2+2fguvw^2-2fhuw^3$$
$$-2ghvw^3-bcu^2w^2-abw^4+2bguw^3-c^2u^2v^2$$
$$-cav^2w^2+2cguv^2w+2cfu^2vw+2afvw^3-4fguvw^2)].$$
$$=4w^2[f^2u^2+g^2v^2+h^2w^2+2chuv+2fguv-2fhuw$$
$$-2ghvw-bcu^2-abw^2+2bguw-cav^2+2afvw-4fguv]$$
$$=4w^2[u^2(f^2-bc)+v^2(g^2-ac)+w^2(h^2-ab)$$
$$+2uv(ch-fg)+2vw(af-gh)+2wu(bg-hf)]$$

$$=-4w^2[u^2(bc-f^2)+v^2(ca-g^2)+w^2(ab-h^2)$$
$$+2vw(gh-af)+2wu(hf-bg)-2uv(fg-ch)]$$
$$=-4w^2[\mathbf{A}u^2+\mathbf{B}v^2+\mathbf{C}w^2+2\mathbf{F}vw+2\mathbf{G}wu+2\mathbf{H}uv]$$
$$=4w^2\,\mathbf{P}^2 \text{ (Art. 7·18).}$$

$\therefore$ (7) becomes

$$\frac{l_1l_2}{bw^2+cv^2-2fvw}=\frac{m_1m_2}{cu^2+aw^2-2guw}=\frac{l_1m_2-l_2m_1}{\pm 2w\mathbf{P}},$$

or,
$$\frac{l_1l_2}{bw^2+cv^2-2fvw}=\frac{m_1m_2}{cu^2+aw^2-2guw}=\frac{n_1n_2}{av^2+bu^2-2huv},$$

(from symmetry.)

$$=\frac{l_1m_2-l_2m_1}{\pm 2w\mathbf{P}}=\frac{m_1n_2-m_2n_1}{\pm 2u\mathbf{P}}=\frac{n_1l_2-n_2l_1}{\pm 2v\mathbf{P}} \quad \ldots(8),$$

(from symmetry).

$$\because \quad \frac{l_1l_2}{bw^2+cv^2-2fvw}=\frac{m_1m_2}{cu^2+aw^2-2guw}=\frac{n_1n_2}{av^2+bu^2-2huv},$$

$$\therefore \text{ each }=\frac{l_1l_2+m_1m_2+n_1n_2}{(b+c)u^2+(c+a)v^2+(a+b)w^2-2fvw-2guw-2huv}, \quad \ldots(9)$$

Also, from (8),

$$\text{each }=\frac{\sqrt{(m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2+(l_1m_2-l_2m_1)^2}}{\pm\sqrt{4\mathbf{P}^2(u^2+v^2+w^2)}} \quad \ldots(10)$$

$\therefore$ from (9) and (10), we have

$$\frac{l_1l_2+m_1m_2+n_1n_2}{(b+c)u^2+(c+a)v^2+(a+b)w^2-2fvw-2guw-2huv}$$
$$=\frac{\sqrt{\Sigma(m_1n_2-m_2n_1)^2}}{\pm 2\mathbf{P}\sqrt{u^2+v^2+w^2}},$$

Let θ be the angle between the lines.

$$\therefore \quad \frac{\cos\theta}{(a+b+c)u^2-(au^2+bv^2+cw^2+2fvw+2guw+2huv)}$$
$$=\frac{\sin\theta}{\pm 2\mathbf{P}\sqrt{u^2+v^2+w^2}},$$

or
$$\frac{\cos\theta}{(a+b+c)u^2-f(u,v,w)}=\frac{\sin\theta}{\pm 2\mathbf{P}\sqrt{u^2+v^2+w^2}}.$$

Hence
$$\tan\theta=\frac{\pm 2\mathbf{P}\sqrt{\mathbf{u}^2+\mathbf{v}^2+\mathbf{w}^2}}{(\mathbf{a}+\mathbf{b}+\mathbf{c})\mathbf{u}^2-\mathbf{f}(\mathbf{u},\mathbf{v},\mathbf{w})}.$$

Cor. 1. Condition of tangency of a plane and a cone.

To find the condition that the plane $ux+vy+wz=0$ may touch the cone $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$.

The plane $ux+vy+wz=0$ touches the given cone if the angle between the lines of section is zero, *i.e.*, if

$$\theta=0, \quad \text{or} \quad \tan\theta=0,$$

or $\quad \mathbf{P}=0$, or, $\mathbf{A}u^2+\mathbf{B}v^2+\mathbf{C}w^2+2\mathbf{F}vw+2\mathbf{G}uw+2\mathbf{H}uv=0$,

which is the required condition.

Cor. 2. Condition of perpendicularity. To find the condition that the lines in which the plane $ux+vy+wz=0$ cuts the cone $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ may be perpendicular.

Let θ be the angle between the lines.

$\because$ they are perpendicular,

$$\therefore \quad \theta=90^\circ, \quad \text{or;} \quad \tan\theta=\infty,$$

or,
$$\frac{\pm 2\mathrm{P}(u^2+v^2+w^2)^{1/2}}{(a+b+c)(u^2+v^2+w^2)-f(u,v,w)}=\infty$$

or,
$$(\mathbf{a}+\mathbf{b}+\mathbf{c})(\mathbf{u}^2+\mathbf{v}^2+\mathbf{w}^2)-\mathbf{f}(\mathbf{u}, \mathbf{v}, \mathbf{w})=0,$$

which is the required condition.

7·20. Reciprocal cones : Def.

Two cones are said to be reciprocal if they are such that each is the locus of the normals drawn through the origin to the tangent plane to the other.

7·21. To find the equation of the cone which is reciprocal to the cone $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$

The equation of the cone is

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \qquad \ldots(1)$$

From Art. 7·19, Cor. 1, the plane

$$ux+vy+wz=0 \qquad \ldots(2)$$

touches (1) if

$$\mathbf{A}u^2+\mathbf{B}v^2+\mathbf{C}w^2+2\mathbf{F}vw+2\mathbf{G}uw+2\mathbf{H}uv=0 \qquad \ldots(3)$$

The direction cosines of the normal to the tangent plane (2) are proportional to u, v, w.

$\therefore$ equations of the normal through (0, 0, 0) are

$$\frac{x}{u}=\frac{y}{v}=\frac{z}{w} \qquad \ldots(4)$$

Eliminating u, v, w between (3) and (4), the locus of the normals is

$$\mathbf{A}x^2+\mathbf{B}y^2+\mathbf{C}z^2+2\mathbf{F}yz+2\mathbf{G}zx+2\mathbf{H}xy=0,$$

which is the required equation of the reciprocal cone of (1).

Note. *Let us start with the equation.*

$$\mathbf{A}x^2+\mathbf{B}y^2+\mathbf{C}z^2+2\mathbf{F}yz+2\mathbf{G}zx+2\mathbf{H}xy=0 \qquad \ldots(1)$$

of the cone. The condition that the plane

$$ux+vy+wz=0 \qquad \ldots(2)$$

touches the cone (1) is

$$\mathbf{A}'u^2+\mathbf{B}'v^2+\mathbf{C}'w^2+2\mathbf{F}'vw+2\mathbf{G}'wu+2\mathbf{H}'uv=0 \qquad \ldots(3)$$

where

$$\mathbf{A}'=\mathbf{BC}-\mathbf{F}^2=(ca-g^2)(ab-h^2)-(gh-af)^2$$
$$=a(abc+2fgh-af^2-bg^2-ch^2)$$
$$=a\mathbf{D},\ \mathbf{B}'=b\mathbf{D},\ \mathbf{C}'=c\mathbf{D},\ \textit{and}$$
$$\mathbf{F}'=\mathbf{GH}-\mathbf{AF}=(hf-bg)(fg-ch)-(bc-f^2)(gh-af)$$
$$=(f^2gh-cfh^2-bfg^2+bcgh-bcgh+abcf+f^2gh-af^3.$$
$$=2f^2gh-cfh^2-bfg^2+abcf-af^3$$
$$=f(abc+2fgh-af^2-bg^2-ch^2)=f\mathbf{D},$$
$$\mathbf{G}'=g\,\mathbf{D}\ \textit{and}\ \mathbf{H}'=h\,\mathbf{D}.$$

$\therefore$ *(3) becomes* $au^2+bv^2+cw^2+2fvw+2gwu+2huv=0.$

This shows that the normal at the vertex to the tangent plane (2)

viz., $\dfrac{x}{u}=\dfrac{y}{v}=\dfrac{z}{w}$ *generates the cone*

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \qquad \ldots(4)$$

which is the original cone $f(x, y, z)=0$.

Hence (1) and (4) are reciprocal cones.

7·22. Three mutually perpendicular generators.

To show that (i) the condition that the cone

$$\mathbf{ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0}$$

has three mutually perpendicular generators is $\mathbf{a+b+c=0}$.

(Bihar, 1961)

and (ii) when this condition is statisfied, the cone possesses an infinite number of sets of three mutually perpendicular generators.

Proof

(*i*) Let $\dfrac{x}{u}=\dfrac{y}{v}=\dfrac{z}{w}$ ...(1) be a generator of the cone

$$f(x, y, z)\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \quad ...(2)$$

$$\therefore \quad f(u, v, w)\equiv au^2+bv^2+cw^2+2fvw+2gwu+2huv=0 \quad ...(3)$$

(Art. 7·5)

The equation of the plane through (0, 0, 0) perpendicular to (1) is $ux+vy+wz=0$ ...(4)

If (4) cuts (2) in two perpendicular generators, then

$$(a+b+c)(u^2+v^2+w^2)-f(u, v, w)=0 \quad \text{[Art. 7·19, cor. 2]}$$

or, $$(a+b+c)(u^2+v^2+w^2)=0, \quad \text{using (3)}$$

$\therefore$ $\mathbf{a+b+c=o}$ ($\because$ $u^2+v^2+w^2\neq 0$), which is the required condition that (2) has three mutually perpendicular generators., *viz.*, the generator (1) and the two perpendicular lines of section of (2) and (4).

(*ii*) $\because$ $\dfrac{x}{u}=\dfrac{y}{v}=\dfrac{z}{w}$ is any arbitrary generator,

$\therefore$ we find that if $a+b+c=0$, then the plane through the vertex (0, 0, 0) perpendicular to any generator of the cone cuts it in two other perpendicular generators. These two generators will themselves be perpendicular to the first generator.

$\therefore$ if $a+b+c=0$, the cone has an **infinite** number of sets of three mutually perpendicular generators. This proves the proposition.

EXAMPLES VII (G)

Type I. Ex. 1. Find the equations to the lines in which the plane $2x+y-z=3$ cuts the cone $4x^2-y^2+3z^2=0$.

(*Karnatak Engg., 1961 ; Pakistan, 1958S*)

Sol. The plane is $2x+y-z=0$ ...(1) and the cone is

$$4x^2-y^2+3z^2=0 \quad ...(2)$$

Let $$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$$

be a line of section of (1) and (2).

$\therefore \quad 2l+m-n=0 \quad ...(3) \quad$ and $4l^2-m^2+3n^2=0 \quad ...(4)$

Eliminating n between (3) and (4), we have

$$4l^2-m^2+3(2l+m)^2=0,$$

or,
$$16l^2+2m^2+12lm=0,$$

or,
$$8l^2+6ml+m^2=0.$$

$$\therefore \quad l=\frac{-6m\pm\sqrt{36m^2-32m^2}}{16}=\frac{-6m\pm 2m}{16}$$

$$\therefore \quad l=-\frac{m}{4},-\frac{m}{2},$$

Now $4l+m=0$
From (3), $2l+m-n=0$.
$\therefore \frac{l}{-1}=\frac{m}{4}=\frac{n}{2}$

$2l+m=0$
From (3) $2l+m-n=0$
$\therefore \quad \frac{l}{-1}=\frac{m}{2}=\frac{n}{0}$.

$\therefore$ direction cosines of the lines of section of (1) and (2) are proportional to $-1, 4, 2$ and $-1, 2, 0$.

$\therefore$ equations of the lines of section are

$$\frac{x}{-1}=\frac{y}{4}=\frac{z}{2} \text{ and } \frac{x}{-1}=\frac{y}{2}=\frac{z}{0}.$$

Ex. 2. Find the equations to the lines in which the plane $3x+4y+z=0$ cuts the cone $15x^2-32y^2-7z^2=0$.

$$\left[\textbf{Ans. } \frac{x}{-3}=\frac{y}{2}=\frac{z}{1}\ ;\ \frac{x}{2}=\frac{y}{-1}=\frac{z}{-2}.\right]$$

Type II. Ex. 1. Find the angles between the lines of section of the plane $3x+y+5z=0$ and the cone $6yz-2zx+5xy=0$.

(Delhi Hons., 1953 ; Pakistan, 1957)

Sol. The plane is $3x+y+5z=0 \quad ...(1)$ and the cone is

$$6yz-2zx+5xy=0 \qquad ...(2)$$

Let the equations of a line of section be

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \qquad ...(3)$$

$\therefore$ (3) lies in (1), $\quad \therefore \quad 3l+m+5n=0 \qquad ...(4)$

$\therefore$ (3) lies on (2), $\quad \therefore \quad 6mn-2nl+5lm=0. \qquad ...(5)$

Eliminating n between (3) and (4), we have

$$6m\left(\frac{-3l-m}{5}\right)-2l\left(\frac{-3l-m}{5}\right)+5lm=0$$

or,
$$6l^2+9lm-6m^2=0,$$

or,
$$2l^2+3lm-2m^2=0$$

$$\therefore \quad l=\frac{3m\pm\sqrt{9m^2+16m^3}}{4}=\frac{-3m\pm 5m}{4}.$$

16470.

$$\therefore \quad l=\frac{m}{2}, -2m.$$

Now $2l-m=0$

From (3), $3l+m+5n=0$

$\therefore \quad \frac{l}{-5}=\frac{m}{-10}=\frac{n}{5}$

or $\quad \frac{l}{-1}=\frac{m}{-2}=\frac{n}{1}$

$l+2m=0$

From (3), $3l+m+5n=0$

$\therefore \quad \frac{l}{10}=\frac{m}{-5}=\frac{n}{-5}$

or $\quad \frac{l}{2}=\frac{m}{-1}=\frac{n}{1}$.

$\therefore$ the direction consines of the lines of section are proportional to $-1, -2, 1$ and $2, -1, 1$.

Let θ be the angle between them.

$$\therefore \quad \cos\theta=\frac{(-1)(2)+(-2)(-1)+(1)(1)}{\sqrt{1^2+2^2+1^2}\ \sqrt{2^2+1^2+1^2}}=\frac{1}{6}$$

$$\therefore \quad \theta=\cos^{-1}\left(\frac{1}{6}\right).$$

Ex. 2. Find the angles between the lines of section of the following planes and cones :

(*i*) $6x-10y-7z=0,\ 108x^2-20y^2-7z^2=0.$ *(Allahabad, 1952)*

(*ii*) $x+3y-2z=0,\ x^2+9y^2-4z^2=0.$

(*iii*) $x-3y+z=0,\ x^2-5y^2+z^2=0.$ *(Calcutta, 1960)*

$\left[\textbf{Ans.}\ (i)\ \cos^{-1}\left(\frac{16}{21}\right),\ (ii)\ \cos^{-1}\left(\frac{3}{\sqrt{65}}\right),\ (iii)\ \cos^{-1}(5/6).\right]$

Ex. 3. Find the equation of the cone whose vertical angle is 90°, which has its vertex at (0, 0, 0) and its axis along the line $x=-2y=z$ and show that the plane $z=0$ cuts the cone in two lines inclined at an angle $\cos^{-1}\frac{4}{5}$.

(A.M.I.E, Nov., 1958 ; Raj. Engg. 1955)

[**Ans.** $x^2+7y^2+z^2+8yz-16zx+8xy=0$.]

Type III. Ex. 1. Prove that the plane $\mathbf{ax+by+cz=0}$ cuts the cone $\mathbf{yz+zx+xy=0}$ in perpendicular lines

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0.$$

(Delhi Hons., 1955, 1959 ; A.M.I.E., 1959 ; Karnatak Engg. 1961 ; Punjab B.Sc., 1961S ; Pakistan, 1955S ; Agra, 1953, 1962 ; Punjab, 1954S ; Aligarh, 1962).

Sol. Let $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ ...(1)

be a line of section of the plane

$$ax+by+cz=0 \quad ...(2)$$

and the cone

$$yz+zx+xy=0. \quad ...(3)$$

$\because$ (1) lies in (2),

$$\therefore \quad al+bm+cn=0 \quad ...(4)$$

$\because$ (1) lies on (3),

$$\therefore \quad mn+nl+lm=0 \quad ...(5)$$

Eliminating n between (4) and (5), we have

$$(l+m)(al+bm)-clm=0,$$

or, $$al^2+(a+b-c)lm+bm^2=0,$$

or, $$a\frac{l^2}{m^2}+(a+b-s)\frac{l}{m}+b=0.$$

Let its roots be $\frac{l_1}{m_1}, \frac{l_2}{m_2}$.

$\therefore$ $$\frac{l_1l_2}{m_1m_2}=\frac{b}{a}.$$

Similarly, $$\frac{m_1m_2}{n_1n_2}=\frac{c}{b}.$$

$\therefore$ $$\frac{l_1l_2}{bc}=\frac{m_1m_2}{ca}=\frac{n_1n_2}{ab}=\lambda, \text{ say.}$$

$\therefore$ $$l_1l_2+m_1m_2+n_1n_2=\lambda(bc+ca+ab).$$

$\because$ the lines of section are perpendicular,

$\therefore$ $$bc+ca+ab=0,$$

or, $$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0.$$

Aliter.

From Art. 7·19, cor. 2, if the given plane cuts the given cone in perpendicular lines, then

$$bc+ca+ab=0, \qquad [(a+b+c)(u^2+v^2+w^2)-f(u, v, w)=0]$$

or, $$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0.$$

Aliter.

$yz+zx+xy=0$ is the equation of a cone in which the coefficients of x^4, y^2, z^2 are each zero.

$\therefore$ sum of the coefficients of x^2, y^2, z^2 is zero.

$\therefore$ it is a cone which has three mutually perpendicular generators (Art. 7·22).

$\therefore$ if the plane $ax+by+cz=0$ cuts the cone in three mutually perpendicular generators, its normal *viz.*,

$$\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$$

will lie on the cone.

$\therefore$ $$bc+ca+ab=0,$$

or, $$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0.$$

Ex. 2. Show that the angle between the lines given by

$$x+y+z=0,$$

$$ayz+bzx+cxy=0$$

is $\frac{\pi}{2}$ if $$a+b+c=0,$$

but $\frac{\pi}{3}$ if $$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0.$$

(*Delhi Hons., 1955 ; Kashmir, 1951 ; Raj. Engg.; 1958 ; Vikram, 1962 ; Sagar, 1962*)

Type IV. Ex. 1. Prove that the cones

$$\mathbf{ax^2+by^2+cz^2=0}$$

and $$\mathbf{\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=0.}$$

are reciprocal. (*Baroda, 1953 ; Delhi Hons., 1957 ; Punjab B.Sc., 1962S*)

Sol. Here, $a\equiv a$, $b\equiv b$, $c\equiv c$, $f=0$, $g=0$, $h=0$.

$\therefore$ $\mathbf{A}=bc-f^2=bc$, $\mathbf{F}=gh-af=0$

$\mathbf{B}=ca-g^2=ca$, $\mathbf{G}=hf-bg=0$

$\mathbf{C}=ab-h^2=ab$; $\mathbf{H}=fg-ch=0$.

$\therefore$ equation of the cone reciprocal to the cone

$$ax^2+by^2+cz^2=0$$

is $$bcx^2+cay^2+abz^2=0,$$

or, $$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=0.$$

Aliter.

The plane $ux+vy+wz=0$ ...(1)

is a tangent plane at the origin to the cone

$$ax^2+by^2+cz^2=0$$

if $$\begin{vmatrix} a & 0 & 0 & u \\ 0 & b & 0 & v \\ 0 & 0 & c & w \\ u & v & w & 0 \end{vmatrix}=0,$$

or if $$a\begin{vmatrix} b & 0 & v \\ 0 & c & w \\ v & w & 0 \end{vmatrix}-u\begin{vmatrix} 0 & b & c \\ 0 & 0 & c \\ u & u & w \end{vmatrix}=0,$$

or if $$bcu^2+cav^2+abw^2=0,$$

or if $$\frac{u^2}{a}+\frac{v^2}{b}+\frac{w^2}{c}=0 \qquad ...(2)$$

The normal to the plane (1) at the origin is

$$\frac{x}{u}=\frac{y}{v}=\frac{z}{w}$$

which is a generator of the cone

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=0,$$

in view of (2).

$\therefore$ the given cones are reciprocal.

Ex. 2. Prove that the tangent planes to the cone $fyz+gzx+hxy=0$ are at right angles to the generators of the cone

$$f^2z^2+g^2y^2+h^2z^2-2ghyz-2hfzx-2fgxy=0. \qquad \textit{(Delhi Hons., 1952)}$$

Ex. 3. Prove that the perpendiculars drawn from the origin to the tangent planes to the cone

$$3x^2+4y^2+5z^2+2yz+4zx+6xy=0$$

lice on the cone

$$19x^2+11y^2+3z^2+6yz-10zx-26xy=0. \qquad \textit{(Kashmir, 1958)}$$

Type V. Ex. 1. Prove that the equation $\sqrt{fx}+\sqrt{gy}+\sqrt{hz}=0$ represents a cone that touches the coordinate planes (*Punjab, 1956*), **andthat the equation of the reciprocal cone is $fyz+gzx+hxy=0$.**

(*Raj., 1960, 1963 ; Jodhpur, 1964 ; Sagar, 1960 ; Agra, 1958*)

Sol. We have, $\sqrt{fx}+\sqrt{gy}+\sqrt{hz}=0$ $\qquad$ ...(1)

or, $\quad (\sqrt{fx}+\sqrt{gy})^2=hz,\quad$ or, $\quad (fx+gy-hz)^2=(-2\sqrt{fgxy})^2,$

or, $\quad f^2x^2+g^2y^2+h^2z^2+2fgxy-2ghyz-2fhxz=4fgxy,$

or, $\quad f^2x^2+g^2y^2+h^2z^2-2ghyz-2fhxz-2fgxy=0.$

$\because$ it is a homogeneous equation of the seconp degree in x, y and z,

$\therefore$ it represents a cone.

Now $\quad A=g^2h^2-g^2h^2=0,\qquad B=h^2f^2-h^2f^2=0,\qquad C=f^2g^2-f^2g^2=0$

$F=f^2gh+f^2gh=2f^2gh,\qquad G=fg^2h+g^2fh=2g^2fh,$

$H=gh^2f+h^2fg=2h^2fg.$

$\therefore$ equation of the cone reciprocal to (1) is

$$2f^2ghyz+2g^2fhzx+2h^2fgxy=0$$

or, $\quad fyz+gzx+hxy=0.$

$\because$ this cone contains three axes as generators,

$\therefore$ the original cone (1) has three coordinate planes as tangent planes.

Ex. 2. Show that the general equation to a cone which touches the coordinate planes is

$$a^2x^2+b^2y^2+c^2z^2-2bcyz-2cazx-2abxy=0.$$

(*Kashmir, 1956 ; Punjab, 1959 ; Punjab B.Sc., 1959 S*)

Ex. 3. Find the equation of the quadric cone which touches the coordinate planes and the three mutually perpendicular planes

$$x-y+z=0, \quad 2x+3y+z=0, \quad 4x-y-5z=0.$$

[**Ans.** $64x^2+9y^2+25z^2-30yz-80zx+48xy=0.$]

Type VI. Ex. 1. (i) Prove that the cone

$$\mathbf{ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0}$$

has three mutually perpendicular tangent planes if

$$\mathbf{bc+ca+ab=f^2+g^2+h^2}.$$

(Raj., 1955, 1962 ; Pakistan Hons., 1953)

(ii) If a right circular cone has three mutually perpendicular generators, the semi-vertical angle is $\tan^{-1}(\sqrt{2})$.

(Punjab, 1956 ; Sagar, 1960 ; Delhi Hons., 1953)

Sol. (*i*) The given cone will have three mutually perpendicular tangent planes if its reciprocal cone has three mutually perpendicular generators. The cone reciprocal to the given cone is

$$\mathbf{A}x^2+\mathbf{B}y^2+\mathbf{C}z^2+2\mathbf{F}yz+2\mathbf{G}zx+2\mathbf{H}xy=0,$$

where
$$\mathbf{A}=bc-f^2, \quad \mathbf{B}=ca-g^2, \quad \mathbf{C}=ab-h^2,$$
$$\mathbf{F}=gh-af, \quad \mathbf{G}=hf-bg, \quad \mathbf{H}=fg-ch.$$

This cone has three mutually perpendicular generators if $A+B+C=0$,

or if
$$bc-f^2+ca-g^2+ab-h^2=0,$$
or,
$$bc+ca+ab=f^2+g^2+h^2.$$

(*ii*) Equation of the right circular cone whose vertex is origin axis is z-axis and the semi-vertical angle is α, is $(x^2+y^2-z^2\tan^2\alpha)=0$.

Here $a=1, \; b=1, \; c=-\tan^2\alpha.$ (Art. 7·9)

$\therefore$ condition for three mutually perpendicular generators is

$$a+b+c=0, \quad \text{or,} \quad 1+1-\tan^2\alpha=0,$$

or,
$$\alpha=\tan^{-1}\sqrt{2}.$$

Ex. 2. Obtain the condition that the plane $lx+my+nz=0$ should cut the cone $(b-c)x^2+(c-a)y^2+(a-b)z^2=0$ in perpendicular lines.

(Karnatak Engg., 1961)

[**Ans.** $(b-c)l^2+(c-a)m^2+(a-b)n^2=0.$]

Ex. 3. If $\dfrac{x}{1}=\dfrac{y}{2}=\dfrac{z}{3}$ *represent one of a set of three mutually perpendicular generators of the cone* $5yz-8zx-3xy=0$, *find the equations to the other two.* *(Delhi, Hons., 1960 ; Punjab, 1954S ; Raj., 1956 ; Raj. Engg., 1963)*

[**Ans.** $x=y=-z, \quad 4x=-5y=20z.$]

[**Hint.** *The plane containing the other two generators is* $x+2y+3z=0$.

Let $\dfrac{x}{l}=\dfrac{y}{m}=\dfrac{z}{n}$ *is a generator lying in this plane.*

$\therefore \quad l+2m+3n=0, \quad 5mn-8nl-3lm=0.$

Eliminate n and get two equations of l and m. Combine each with $l+2m+3n=0$ *and get the proportional values of l, m, n*].

Type VII. Ex. 1. Find the locus of points from which three mutually perpendicular lines can be drawn to intersect the conic $ax^2+by^2=1$, $z=0$. *(Karnatak Engg., 1961 ; Agra, 1951)*

Sol. Let the point from which three mutually perpendicular lines can be drawn to intersect the conic $ax^2+by^2=1$, $z=0$. ...(1)
be (λ, μ, ν). Equation of a line through (λ, μ, ν) is

$$\frac{x-\lambda}{l}=\frac{y-\mu}{m}=z-\nu. \quad ...(2)$$

This line meets $z=0$ in $(\lambda-l\nu, \mu-m\nu, 0)$.

This point lies on the conic $ax^2+by^2=1$

if $$a(\lambda-l\nu)^2+b(\mu-m\nu)^2=1 \quad ...(3)$$

Eliminating l, m between (2) and (3). we have

$$a\left[\lambda-\nu.\frac{x-\alpha}{z-\nu}\right]^2+b\left[\mu-\nu\frac{(y-\beta)}{z-\nu}\right]^2=1,$$

or, $$a(\lambda z-\nu x)^2+b(\mu z-\nu y)^2-(z-\nu)^2=0$$

or, $$z^2(a\lambda^2+b\mu^2-1)-2z\nu(a\lambda x+b\mu\nu-1)+\nu^2(ax^2+by^2-1)=0.$$

The required locus is obtained by equating the sum of the coefficients of x^2, y^2 and z^2 to zero.

$\therefore$ $a\nu^2+b\nu^2+a\lambda^2+b\mu^2-1=0,$

$\therefore$ locus of (λ, μ, ν) is $ax^2+by^2+(a+b)z^2=1$.

Ex. 2. Show that the locus of points from which three mutually perpendicular lines can be drawn to intersect a given circle is a surface of revolution.

Ex. 3. Prove that the locus of points from which three mutually perpendicular planes can be drawn to touch the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z=0,$$

is the sphere $x^2+y^2+z^2=a^2+b^2$.

MISCELLANEOUS (REVISION) EXAMPLES ON CHAPTER VII

1. Planes through OX and OY include an angle α. Show that their line of intersection lies on the cone $z^2(x^2+y^2+z^2)=x^2y^2\tan^2\alpha$.
(Punjab, 1955 ; Agra, 1954)

2. The vertex of a cone is (a, b, c) and the yz-plane cuts it in the curve $F(y, z)=0$, $x=0$. Show that the zx-plane cuts it in the curve

$$y=0,\ F\left(\frac{bx}{x-a}, \frac{cx-az}{x-a}\right)=0. \quad \textit{(Raj., 1950)}$$

Sol. Let $$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n} \quad ...(1)$$

be a generator of the cone with vertex at (a, b, c).

It meets $x=0,$

where $$y = b - \frac{ma}{l}$$

and $$z = c - \frac{na}{l}.$$

$\because$ (1) meets $F(y, z)=0,\ x=0,$

$$\therefore \qquad F\left[b - \frac{ma}{l},\ c - \frac{na}{l}\right]=0.$$

$\therefore$ locus of (1) is

$$F\left[b - a.\left(\frac{y-b}{x-a}\right),\ c-a.\left(\frac{z-c}{x-a}\right)\right]=0,$$

or, $$F\left[\frac{bx-ay}{x-a},\ \frac{cx-az}{x-a}\right]=0.$$

Its curve of intersection with the plane $y=0$ is

$$y=0,\ F\left[\frac{bx}{x-a},\ \frac{cx-az}{y-a}\right]=0.$$

3. *Prove that the equation to the planes through the origin perpendicular to the lines of section of . ... $x+my+nz=0$ and the cone $ax^2+by^2+cz^2=0$ is $x(bn^2+cm^2)+y^2(cl^2+an^2)+z^2(am^2+bl^2)-2amnyz-2bnlzx-2clmxy=0$.* *(Raj., 1952)*

Sol. Let l_1, m_1, n_1 ; l_2, m_2, n_2 be the direction cosines of the lines of section of the plane $$lx+my+nz=0 \qquad ...(1)$$
and the cone $$ax^2+by^2+cz^2=0 \qquad ...(2)$$

$\therefore$ required planes are
$$l_1x+m_1y+n_1z=0,\ l_2x+m_2y+n_2z=0.$$

The equations of the lines of section are
$$\frac{x}{l_1}=\frac{y}{m_1}=\frac{z}{n_1} \quad \text{and} \quad \frac{x}{l_2}=\frac{y}{m_2}=\frac{z}{n_2}.$$

The combined equation to the planes is
$$(l_1x+m_1y+n_1z)(l_2x+m_2y+n_2z)=0,$$
or, $$\Sigma l_1l_2x^2+\Sigma(l_1m_2+l_2m_1)xy=0 \qquad ...(1)$$

From Art. 7·19, we have
$$\frac{l_1l_2}{bn^2+cl^2}=\frac{m_1m_2}{cl^2+an^2}=\frac{n_1n_2}{am^2+\ l^2}=\frac{l_1m_2+l_2m_1}{-2clm}$$
$$=\frac{m_1n_2+m_2n_1}{-2amn}=\frac{n_1l_2+n_2l_1}{-2bnl}.$$

$\therefore$ (1) becomes $\Sigma\,(bn^2+cm^2)x^2-\Sigma 2amnyz=0.$

4. *If a line OP, drawn through the vertex O of the cone $ax^2+by^2+cz^2=0$, is such that the two planes through OP, each of which cuts the cone in a pair of perpendicular lines, are at right angles, prove that the locus of OP is the cone*
$$(2a+b+c)x^2+(2b+c+a)y^2+(2c+a+b)z^2=0.$$
(Punjab Hons., 1961)

Sol. Let OP be the line

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \qquad ...(1)$$

and let
$$ux+vy+wz=0 \qquad ..(2)$$
be a plane through OP cutting the cone
$$ax^2+by^2+cz^2=0 \qquad ...(3)$$
at right angles.

$\therefore$ From Cor. 2, Art. 7·19, we have
$$(a+b+c)(u^2+v^2+w^2)=au^2+bv^2+cw^2,$$
or,
$$(b+c)u^2+(c+a)v^2+(a+b)w^2=0 \qquad ...(4)$$

$\because$ (1) lies in (2),

$\therefore$
$$lu+mv+nw=0 \qquad ...(5)$$

Eliminating w between (1) and (2), we have
$$(b+c)u^2+(c+a)v^2+(a+b)\left[\frac{lu+mv}{-n}\right]^2=0,$$
or
$$[(b+c)n^2+(a+b)l^2]u^2+2uv(a+b)lm+[(a+b)m^2+(c+a)n^2]v^2=0,$$
or,
$$[(b+c)n^2+(a+b)l^2]\frac{u^2}{v^2}+2\frac{u}{v}(a+b)lm+[(a+b)m^2+(c+a)n^2]=0 \qquad ...(6)$$

Let
$$u_1x+v_1y+w_1z=$$
and
$$u_2x+v_2y+w_2z=0$$
be the two planes of the form (2).

$\therefore$ $\frac{u_1}{v_1}$ and $\frac{u_2}{v_2}$ are the roots of (6).

$\therefore$
$$\frac{u_1u_2}{v_1v_2}=\frac{(a+b)m^2+(c+a)n^2}{(b+c)n^2+(a+b)l^2},$$
or,
$$\frac{u_1u_2}{(a+b)m^2+(c+a)n^2}=\frac{v_1v_2}{(a+b)l^2+(b+c).n^2}=\frac{w_1w_2}{(b+c)m^2+(c+a)\ l^2},$$
due to symmetry,

$\because$ the planes are at right angles,

$\therefore$
$$u_1u_2+v_1v_2+w_1w_2=0,$$
or
$$\Sigma l^2(2a+b+c)=0.$$

$\therefore$ locus of OP is $\Sigma x^2(2a+b+c)=0$.

5. Show that the plane $ax+by+cz=0$ cuts the cone $yz+zx+xy=0$ in two lines inclined at an angle
$$\tan^{-1}\left[\frac{\{(a^2+b^2+c^2)(a^2+b^2+c^2-2bc-2ca-2ab)\}^{\frac{1}{2}}}{bc+ca+ab}\right],$$
(Delhi Hons., 1958 ; Punjab Hons.,

and by considering the value of this expression when $a+b+c=0$, show that the cone is of revolution and that its axis is $x=y=z$ and vertical angle is $\tan^{-1}(2\sqrt{2})$. *(Punjab Hons., 1958 ; Agra, 1956)*

6. **Prove that the common generators of the cones** $(\mathbf{b^2c^2-a^4})\,x^2+(\mathbf{c^2a^2-b^4})\,y^2+(\mathbf{a^2b^2-c^4})\,\mathbf{z}^2=0$ **and**
$$\frac{\mathbf{bc-a^2}}{\mathbf{ax}}+\frac{\mathbf{ca-b^2}}{\mathbf{by}}+\frac{\mathbf{ab-c^2}}{\mathbf{cz}}=0 \text{ lie in the planes}$$
$$(\mathbf{bc\pm a^2})\mathbf{x}+(\mathbf{ca\pm b^2})\,\mathbf{y}+(\mathbf{ab\pm c^2})\,\mathbf{z}=0.$$

Sol. Let $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ ...(1) be a common generator of the given cones.

$\therefore$ $(b^2c^2-a^4)l^2+(c^2a^2-b^4)m^2+(a^2b^2-c^4)n^2=0$ ...(1)

and $bcmn(bc-a^2)+canl(ca-b^2)+ablm(ab-c^2)=0$ ...(2)

Multiplying (2) by 2 and subtracting from (1),

we have $(bc+a^2)(bc-a^2)l^2+(ca+b^2)(ca-b^2)m^2+(ab+c^2)(ab-c^2)n^2$

$-2bcmn(bc-a^2)-2canl(ca-b^2)-2ablm(ab-c^2)=0,$

or, $[(bc+a^2)l+(ca+b^2)m+(ab+c^2)n]\cdot[(bc-a^2)l+(ca-b^2)m+(ab-c^2)n]=0,$

$\therefore$ Common generators of the given cones lie on the planes

$$(bc+a^2)x+(ca+b^2)y+(ab+c^2)z=0$$

and $(bc-a^2)x+(ca-b^2)y+(ab-c^2)z=0,$

i.e., lie on the planes $(bc\pm a^2)x+(ca\pm b^2)y+(ab\pm c^2)z=0.$

7. OP and OQ are two straight lines that remain at right angles and move so that the plane OPQ always passes through the z-axis. If OP describes a cone $F\left(\frac{y}{x},\frac{z}{x}\right)=0,$

prove that OQ describes the cone

$$F\left[\frac{y}{x},\left(-\frac{x}{z}-\frac{y^2}{zx}\right)\right]=0.$$

[**Hint.** *The plane OPQ contains OZ,* aed $\therefore$ *it is perpendicular to the plane XOY. Let its equation be* $ax+by=0$.

Let OP be $\frac{x}{l_1}=\frac{y}{m_1}=\frac{z}{n_1}$

and OQ be $\frac{x}{l_2}=\frac{y}{m_2}=\frac{z}{n_2}.$

$\because$ *OP is a generator of the cone* $F\left(\frac{y}{x},\frac{z}{x}\right)=0,$

$\therefore$ $\left(\frac{m_1}{l_1},\frac{n_1}{l_1}\right)=0$ ...(1)

$\because$ *OP, OQ lie in the plane* $ax+by=0,$

$\therefore$ $al_1+bm_1=0,\quad al_2+bm_2=0.$

$\therefore$ $\frac{m_1}{l_1}=\frac{m_2}{l_2}$ ...(2)

$\because$ *OP, OQ are at right angles,* $\therefore$ $l_1l_2+m_1m_2+n_1n_2=0.$

or, $l_2+\frac{m_1}{l_1}m_2+\frac{n_1}{l_1}n_2=0.$

$\therefore$ $\frac{n_1}{l_1}=-\frac{l_2}{n_2}-\frac{m_2^2}{m_2l_2}$, *using* (2).

$\therefore$ (*1*) *becomes* $F\left[\frac{m_2}{l_2},\left\{-\frac{l_2}{n_2}-\frac{m_2^2}{l_2n_2}\right\}\right]=0$

8

The Cylinder

8·1. Cylinder : Def.

The cylinder is the surface generated by a variable straight line (called the generator) which remains parallel to a fixed straight line (called the axis) and satisfies one more condition, for example, it may intersect a given curve (called the guiding curve) or it may touch a given surface.

Note. *Any straight line lying on the cylinder is called its generator.*

8·2. Right circular cylinder : Def.

The right circular cylinder is the surface generated by a straight line (called the generator) which is parallel to a fixed line (called the axis) and is at a constant distance (called the radius) from it.

8·3. To find the equation of a cylinder whose generators are parallel to the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ and base the conic $ax^2+by^2+2hxy+2gx+2fy+c=0,\ z=0,$

The given base is

$$ax^2+2hxy+by^2+2gz+2fy+c=0 \qquad \ldots(1)$$

$$z=0,$$

and the given line is

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \qquad \ldots(2)$$

Equations of a generator parallel to (2) are

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n},$$

where (x, y, z) is **any** point on this generator.

It meets $$z=0,$$

where $$\frac{x-x_1}{l}=\frac{y-y_1}{m}=-\frac{z_1}{n},$$

or, $$x=x_1-\frac{l}{n}z_1,$$

$$y=y_1-\frac{m}{n}z_1.$$

Substituting these values of x and y in (1), we have

$$a\left(x_1-\frac{l}{n}z_1\right)^2+2h\left(x_1-\frac{l}{n}z_1\right)\left(y_1-\frac{m}{n}z_1\right)+b\left(y_1-\frac{m}{n}z_1\right)^2$$
$$+2g\left(x_1-\frac{l}{n}z_1\right)+2f\left(y_1-\frac{m}{n}z_1\right)+c=0$$

$\therefore$ locus of (x_1, y_1, z_1) is

$$a\left(x-\frac{l}{n}z\right)^2+2h\left(x-\frac{l}{n}z\right)\left(y-\frac{m}{n}z\right)+b\left(y-\frac{m}{n}z\right)^2$$
$$+2g\left(x-\frac{l}{n}z\right)+2f\left(y-\frac{m}{n}z\right)+c=0.$$

or, $a(nx-lz)^2+2h(nx-lz)(ny-mz)+b(ny-mz)^2+2gn(nx-lz)$
$$+2fn(ny-mz)+cn^2=0,$$
which is the required equation of the cylinder.

Cor. If the generators are parallel to the z-axis,

then $$l=0=m$$
and $$n=1.$$

$\therefore$ the equation of the cylinder becomes
$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

An observation. *This equation represents a conic in two dimensions and represents a cylinder whose generators are parallel to the z-axis in three dimensions. In general, the equation $f(x, y)=0$ represents a cylinder passing through the curve $f(x, y)=0$, $z=0$ and with generators parallel to the z-axis.*

8·4. Equation of right circular cylinder.

To find the equation of right circular cylinder whose axis is the line

$$\frac{\mathbf{x}-\alpha}{\mathbf{l}}=\frac{\mathbf{y}-\beta}{\mathbf{m}}=\frac{\mathbf{z}-\gamma}{\mathbf{n}},$$

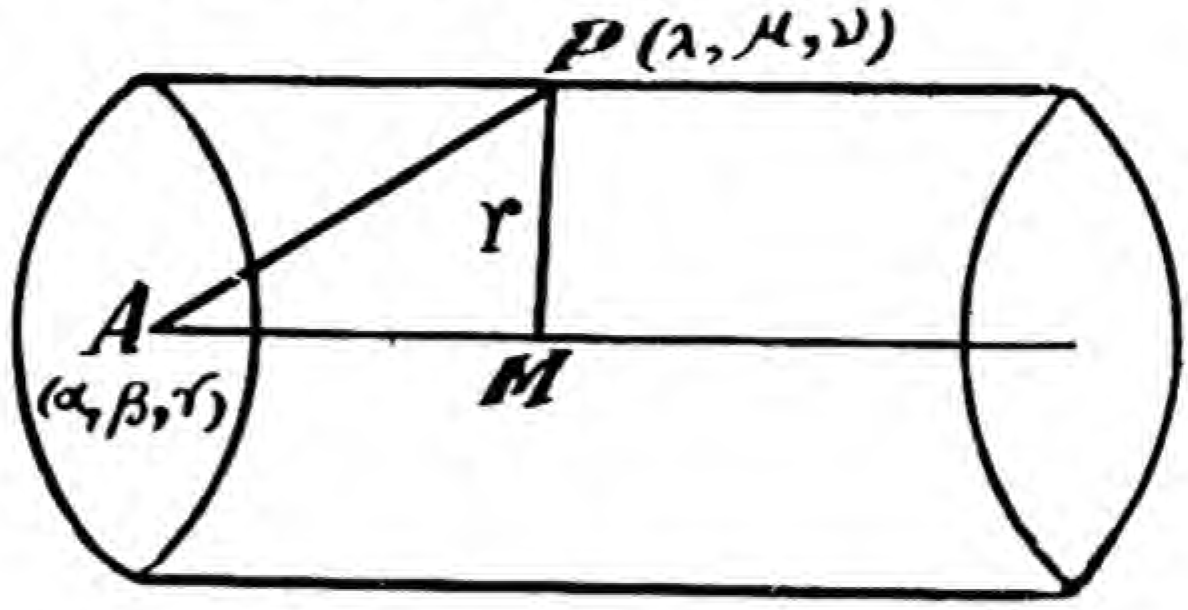
and whose radius is a.

The equations of the axis are
$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \quad \ldots(1)$$

Let $P(\lambda, \mu, \nu)$ be **any** point on the cylinder.

Draw PM perpendicular to the axis (1).

Join P to the point $A(\alpha, \beta, \gamma)$.



Now $PM = r$, and $PA = \sqrt{(\lambda-\alpha)^2 + (\mu-\beta)^2 + (\nu-\gamma)^2}$

Also, AM = Projection of AP on the line whose direction cosines are

$$\frac{l}{\sqrt{l^2+m^2+n^2}}, \frac{m}{\sqrt{l^2+m^2+n^2}}, \frac{n}{\sqrt{l^2+m^2+n^2}}$$

$$= \frac{(\lambda-\alpha)l + (\mu-\beta)m + (\nu-\gamma)n}{\sqrt{l^2+m^2+n^2}}$$

From the right-angled triangle AMP, we have

$AP^2 = AM^2 + PM^2$,

or, $$(\lambda-\alpha)^2 + (\mu-\beta)^2 + (\nu-\gamma)^2 = \frac{[l(\lambda-l) + m(\mu-\beta) + n(\nu-\gamma)]^2}{l^2+m^2+n^2} + r^2$$

$\therefore$ locus of (λ, μ, ν) is

$$(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 = \frac{[l(x-\alpha) + m(y-\beta) + n(z-\gamma)]^2}{l^2+m^2+n^2} + r^2.$$

or, $$(\mathbf{l}^2 + \mathbf{m}^2 + \mathbf{n}^2)[(\mathbf{x}-\alpha)^2 + (\mathbf{y}-\beta)^2 + (\mathbf{z}-\gamma)^2$$
$$- [\mathbf{l}(\mathbf{x}-\alpha) + \mathbf{m}(\mathbf{y}-\beta) + \mathbf{n}(\mathbf{z}-\gamma)]^2 = (\mathbf{l}^2 + \mathbf{m}^2 + \mathbf{n}^2)r^2,$$

which is the required equation.

8·5. Enveloping cylinder of a sphere : Def.

The locus of the tangent lines drawn to a sphere, parallel to a given line, is called an **enveloping cylinder** of the sphere.

8·6. Equation of the enveloping cylinder of a sphere. To find the equation of the enveloping cylinder of the sphere

$$\mathbf{x}^2 + \mathbf{y}^2 + \mathbf{z}^2 = \mathbf{a}^2$$

whose generators are parallel to the line

$$\frac{\mathbf{x}}{\mathbf{l}} = \frac{\mathbf{y}}{\mathbf{m}} = \frac{\mathbf{z}}{\mathbf{n}}$$

Let (λ, μ, ν) be **any** point on a tangent line to the sphere

$$x^2+y^2+z^2=a^2 \qquad \ldots(1)$$

parallel to the line

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \qquad \ldots(2)$$

The equations of the tangent line are

$$\frac{x-\lambda}{l}=\frac{y-\mu}{m}=\frac{z-\nu}{n} \qquad \ldots(3)$$

Any point on (3) is $\lambda+lr, \mu+mr, \nu+nr)$.

If it lies on the sphere (1), then

$$(\lambda+lr)^2+(\mu+mr)^2+(\nu+nr)^2=a^2,$$

or, $$r^2(l^2+m^2+n^2)+2r[l\lambda+m\mu+n\nu)+(\lambda^2+\mu^2+\nu^2-a^2)=0 \qquad \ldots(4)$$

This is quadratic equation in r.

$\because$ the line (3) touches (1),

$\therefore$ (4) has equal roots, the condition for which is

$$4(l\lambda+m\mu+n\nu)^2=4(l^2+m^2+n^2)(\lambda^2+\mu^2+\nu^2-a^2)$$

$\therefore$ locus of (λ, μ, ν) is

$$(\mathbf{x}^2+\mathbf{y}^2+\mathbf{z}^2-\mathbf{a}^2)(\mathbf{l}^2+\mathbf{m}^2+\mathbf{n}^2)=(\mathbf{lx}+\mathbf{my}+\mathbf{nz})^2,$$

which is the required equation of the enveloping cylinder.

EXAMPLES VIII

Type I. Ex. 1. Find the equation of the circular cylinder whose generating lines have the direction cosines l, m, n and which passes through the circumstance of the fixed circle $x^2+y^2=a^2$ in the ZOX plane.

(A.M.I.E., May, 1956)

Sol. Let P (λ, μ, ν) be **any** point on the cylinder. Equations of the generator through P are

$$\frac{x-\lambda}{l}=\frac{y-\mu}{m}=\frac{z-\nu}{n}.$$

This meets $y=0$,

where $$\frac{x-\lambda}{l}=\frac{z-\nu}{n}=\frac{-\mu}{m},$$

or $$x=\lambda-\frac{l\mu}{m},\ z=\nu-\frac{n\mu}{m}.$$

Substituting these values in the curve $x^2+y^2=a^2$, we have

$$\left(\lambda-\frac{l\mu}{m}\right)^2+\left(\nu-\frac{n\mu}{m}\right)^2=a^2,$$

or $$(m\lambda-l\mu)^2+(m\nu-n\mu)^2=m^2a^2.$$

$\therefore$ locus of (λ, μ, ν) is

$$(mx-ly)^2+(mz-ny)^2=m^2a^2,$$

which is the required equation.

Ex. 2. Find the equation of a cylinder whose generators are parallel to the line $x=-\dfrac{y}{2}=\dfrac{z}{3}$ and whose guiding curve is the ellipse $x^2+2y^2=1$, $z=3$. *(Kashmir, 1955)*

[**Ans.** $3x^2+6y^2+3z^2+8yz-2zx+6x-24y-18z+24=0$.]

Ex. 3. Show that the equation of the cylinder whose generators are parallel to the z-axis and intersects the curve $ax^2+by^2=2z$, $lx+my+nz=p$ is $n(ax^2+by^2)+2(lx+my)=2p$.

Type II. Ex. 1. Find the equation of a right circular cylinder whose axis is $x=2y=-z$ and radius 4, *(Punjab, 1957 S)* **and p[illegible] area of the section of the cylinder by the plane XOY is 24π.** *(Vikram Engg., 1960 ; Karnatak, 1959)*

Sol. Let $P(\lambda, \mu, \nu)$ be any point on the cylinder.

The axis is given to be

$$\frac{x}{1}=\frac{y}{\frac{1}{2}}=\frac{z}{-1}.$$

It passes through the origin O (0, 0, 0).

Draw PM perpendicular to the axis.

Join OP.

Now, $\quad PM=4, \quad OP=\sqrt{\lambda^2+\mu^2+\nu^2}$.

Also, $\quad$ OM = projection of OP on the axis

$$=\lambda \cdot \frac{1}{\sqrt{1^2+\frac{1}{4}+1^2}}+\frac{\mu}{2}\cdot\frac{1}{\sqrt{1^2+\frac{1}{4}+1^2}}-\nu\cdot\frac{1}{\sqrt{1^2+\frac{1}{4}+1^2}}$$

$$=\frac{2\lambda}{3}+\frac{1}{3}\mu-\frac{2}{3}\nu.$$

$$OP^2=OM^2+MP^2,$$

or,
$$\lambda^2+\mu^2+\nu^2=\frac{1}{9}(2\lambda+\mu-2\nu)^2+16.$$

$\therefore$ locus of (λ, μ, ν) is

$$9(x^2+y^2+z^2)=(2x+y-2z)^2+144,$$

or,
$$5x^2+8y^2+5z^2-4xy+4yz+8zx-144=0, \qquad \ldots(1)$$

which is the required equation.

The equation of the plane through the origin perpendicular to the axis is $2x+y-2z=0$. The area of the section of the cylinder by this plane $=\pi(4)^2=16\pi$. Let A be the required area of the section of the cylinder by the plane $z=0$.

Then the above circle is the projection of this section on the plane

$$2x+y-2z=0.$$

Let θ be the angle between this plane and $z=0$ plane.

$$\therefore \quad A\cos\theta=16\pi, \quad \text{or,} \quad A=16\pi\cdot\frac{3}{2} \qquad \left(\because \cos\theta=\frac{2}{3}\right)$$

or,
$$A=24\pi.$$

Ex. 2. The radius of a normal section of a right circular cylinder is 2 units, the axis lies along the line $\frac{x-1}{2}=\frac{y+3}{-1}=\frac{z-2}{5}$, find its equation.

[*A.M.I.E., May, 1961*]

[**Ans.** $29x^2+29y^2+29z^2+2xy-2xz+2yz-30x+150y-93z+75=0.$]

Ex. 3. Find the equation of a right circular cylinder (*i*) of radius 2 whose axis phases through (1, 2, 3) and has direction cosines proportional to 2, −3, 6,

(*A.M.I.E., Nov., 1962 ; Raj. Engg., 1959 ; Gujarat, 1955 ; Jodhpur, Engg., 1965*)

(*ii*) whose radius is 2, whose axis passes through the point (1, 2, 3) and is parallel to the x-axis. (*North Bengal, 1964*)

[**Ans.** (*i*) $45x^2+40y^2+13z^2+36yz-24zx+12xy-42x-280y-126z+294=0$;

(*ii*) $y^2+z^2-4y-6z+9=0.$]

Ex. 4. Find the cartesian equation of the right circular cylinder, whose axis is the z-axis and radius a. (*Pakistan, 1956 S*)

[**Ans.** $x^2+y^2=a^2,\ z=0$]

Ex. 5. Find the equation of the right circular cylinder of radius 2 whose axis is the line $\frac{x-1}{2}=y-2=\frac{z-3}{2}$. (*Kashmir, 1952*)

[**Ans.** $5x^2+8y^2+5z^2-4yz-8zx-4xy+22x-16y-14z-10=0.$]

Type III. Ex. 1. Obtain the equation of a right circular cylinder on the circle through the points (a, o, o), (o, b, o) and (o, o, c) as the guiding curve. What is the equation of the axis ? (*Raj., 1958*)

Sol. The equation of the circle through three given points can be written as

$$x^2+y^2+z^2-ax-by-cz=0 \quad \text{...(1)}$$

and

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \quad \text{...(2)}$$

Let (α, β, γ) be **any** point on the cylinder having the above circle as base.

Equations of a generator through (α, β, γ) and parallel to the axis which is perpendicular to the plane (2 so that its direction cosines are proportional to $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are

$$\frac{x-\alpha}{1/a}=\frac{y-\beta}{1/b}=\frac{z-\gamma}{1/c}.$$

Any point on it is $\left(\alpha+\frac{r}{a}, \beta+\frac{r}{b}, \gamma+\frac{r}{c}\right)$.

$\because$ it meets the circle, $\therefore$ the point will satisfy both (1) and (2).

$$\therefore \quad \left(\alpha+\frac{r}{a}\right)^2+\left(\beta+\frac{r}{b}\right)^2+\left(\gamma+\frac{r}{c}\right)^2$$

$$-a\left(\alpha+\frac{r}{a}\right)-b\left(\beta+\frac{r}{b}\right)-c\left(\gamma+\frac{r}{c}\right)=0.$$

and $$\frac{1}{a}\left(\alpha+\frac{r}{a}\right)+\frac{1}{b}\left(\beta+\frac{r}{b}\right)+\frac{1}{c}\left(\gamma+\frac{r}{c}\right)=1,$$

or, $$r^2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)+2r\left(\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}-\frac{3}{2}\right)$$
$$+(\alpha^2+\beta^2+\gamma^2-a\alpha-b\beta-c\gamma)=0 \qquad ...(3)$$

and $$r=-\left(\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}-1\right)\Big/\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) \qquad ...(4)$$

Eliminating r between (3) and (4), we have

$$\frac{\left(\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}-1\right)^2}{\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}-2\left(\frac{\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}-1}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}\right)\left(\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}-\frac{3}{2}\right)$$
$$+(\alpha^2+\beta^2+\gamma^2-a\alpha-b\beta-c\gamma)=0,$$

or, $$\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)(\alpha^2+\beta^2+\gamma^2-a\alpha-b\beta-c\gamma)$$
$$=\left(\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}-1\right)\left[\left(\frac{2\alpha}{a}+\frac{2\beta}{b}+\frac{2\gamma}{c}-3\right)-\left(\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}-1\right)\right]$$

$\therefore$ locus of (α, β, γ) is

$$(a^{-2}+b^{-2}+c^{-2})(x^2+y^2+z^2-ax-by-cz)$$
$$=\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}-1\right)\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}-2\right),$$

which is the required equation of the cylinder.

Now, axis of the cylinder is a straight line through the centre of the sphere *viz.*, $\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$

and having the direction cosines proportional to $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$.

$\therefore$ its equation is $\dfrac{x-a/2}{a^{-1}}=\dfrac{y-b/2}{b^{-1}}=\dfrac{z-c/2}{c^{-1}}$.

Ex. 2. Find the equation of the right circular cylinder whose guiding circle is $x^2+y^2+z^2=9,\ x-y+z=3.$

(*Kashmir, 1954* ; *Punjab, 1959S* ; *Karnatak, 1953*)

[**Ans.** $x^2+y^2+z^2+yz+xy-zx=9.$]

Ex. 3. Show that the equation of the right circular cylinder, described on the circle through the three points (1, 0, 0), (0, 1, 0) and (0, 0, 1) as the guiding curve, is $x^2+y^2+z^2-yz-zx-xy=1$.

Type IV. Ex. 1. Find the equation of the enveloping cylinder of the sphere $x^2+y^2+z^2-2x+4y=1$, having its generators parallel to the line $x=y=z$. (*Punjab, 1960S* ; *Kashmir, 1953*)

Sol. Let (α, β, γ) be any point on a tangent line parallel to the line

$$\frac{x}{1}=\frac{y}{1}=\frac{z}{1} \qquad \ldots(1)$$

Then the equations of the tangent line are

$$\frac{x-\alpha}{1}=\frac{y-\beta}{1}=\frac{z-\gamma}{1}.$$

Any point on this line is

$$(\alpha+r, \beta+r, \gamma+r).$$

If it lies on the sphere $x^2+y^2+z^2-2x+4y-1=0$ ...(2)

then $(\alpha+r)^2+(\beta+r)^2+(\gamma+r)^2-2(\alpha+r)+4(\beta+r)-1=0,$

or, $3r^2+2\alpha r+2\beta r+2\gamma r-2\alpha-2r+4\beta+4r-1+\alpha^2+\beta^2+\gamma^2=0,$

or, $3r^2+r(2\alpha+2\beta+2\gamma+2)+(4\beta-2\alpha-1+\alpha^2+\beta^2+\gamma^2)=0$ ...(3)

$\because$ the line touches the sphere, $\therefore$ (3) has equal roots, the condition for which is

$$(2\alpha+2\beta+2\gamma+2)^2=12(4\beta-2\alpha-1+\alpha^2+\beta^2+\gamma^2)$$

or $$(\alpha+\beta+\gamma+1)^2=3(4\beta-2\alpha-1+\alpha^2+\beta^2+\gamma^2).$$

$\therefore$ locus of (α, β, γ) is

$$(x+y+z+1)^2=3(4y-2x-1+x^2+y^2+z^2)$$

or $$x^2+y^2+z^2+1+2xy+2z+2xz+2yz+2x+2y$$
$$=12y-6x-3+3x^2+3y^2+3z^2,$$

or, $2x^2+2y^2+2z^2-2xy-2xz-2yz-8x+10y-2z-4=0,$

or, $x^2+y^2+z^2-yz-zx-xy-4x+5y-z=2.$

Ex. 2. Find the equation of the right cylinder which envelops a sphere of centre (a, b, c) and radius r, and has its generators parallel to the direction l, m, n. [**Ans.** $(l^2+m^2+n^2)\{(x-a)^2+(y-b)+(z-c)^2-r^2\}$
$=\{l(x-a)+m(y-b)+n(z-c)\}^2.$]

Ex. 3. Find the enveloping cylinder of the sphere

$$x^2+y^2+z^2-2y-4z=11$$

having its generators parallel to the line $x=-2y=2z$.

[**Ans.** $5x^2+5y^2+5z^2+2xy+2yz-2zx+2x-14y-22z=67.$]

MISCELLANEOUS (REVISION) EXAMPLES ON CHAPTER VIII

1. Show that the coordinates of the foot of the perpendicular from a point $P(\alpha, \beta, \gamma)$ on the line $x=y=-z$ are

$$[\tfrac{1}{3}(\alpha+\beta-\gamma), \tfrac{1}{3}(\alpha+\beta-\gamma), -\tfrac{1}{3}(\alpha+\beta-\gamma)].$$

Deduce the equation of the right circular cylinder of radius a having its axis along the line given above. *(A.M.I.E., 1962)*

[**Ans.** $2x^2+2y^2+2z^2-2xy+2yz+2zx=3a^2.$]

2. A right circular cylinder is cut by a sphere whose centre is on one of the generators of the cylinder. Prove that the projection of the curve of intersection on the plane containing the axis of the cylinder and the centre of the sphere is a parabola whose latus rectum is twice the radius of the cylinder.

(Raj. Engg., 1952)

3. Find the radius of the circle $2x-y-2z+13=0$, $x^2+y^2+z^2=2x+4y+4z+1$, and the equation of the right circular cylinder which has the circle for a normal section.

[**Ans** 1, $5x^2+8y^2+5z^2-4yz+8zx+4xy-34x-28y-20z+56=0$.]

4. Find the equations of the projections upon each of the coordinate planes of the curve of intersection of the plane $x-y+2z-4=0$ and the surface $x^2-yz+3x=0$. *(Calcutta, 1962)*

[**Ans.** $2x^2-y^2+xy+6x-4y=0$, $z=0$; $x^2+2z^2+zx+3x-4z=0$, $y=0$; $y^2+4z^2-5yz+7y-14z+28=0$, $x=0$].

9

The Central Conicoid

9·1. Conicoid or quadric : Def.

The general equation of the second degree in x, y, z, *viz.*,

$F(x, y, z) \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$ represents a locus called a **conicoid** or a **quadric**.

Note. *The equation of the conicoid contains only nine disposable constants.*

We can reduce this equation to some standard form by the suitable change o the axes.

9·2. Centre : Def.

The centre of the conicoid is a point such that every line through it intersects the conicoid in pair of points equidistant from it.

9·3. Central conicoid : Def.

The conicoid having a centre is called a **central conicoid.**

SECTION I

9·4. Tracing of the loci of some standard central conicoids.

[**A note on symmetry.**

(*i*) *If on changing x to $-x$, y to $-y$ and z to $-z$, the equation remains unchanged, i.e., if only even powers of x, y and z occur in the equation of a surface, then the origin is a centre of symmetry.*

(*ii*) *If on changing x to $-x$ and y to $-y$ the equation remains unchanged, i.e., if only even powers of x and y and products of x and y of the same power occur in the equation of a surface, the surface is symmetrical about the z-axis.*

Similar results hold for the symmetry about the x and y axes.

(*iii*) *If on changing x to $-x$, the equation remains unaltered, i.e., if only even powers of x occur in the equation of the surface, the surface is symmetrical about the yz-plane.*

Similar results hold for the symmetry about the zx and xy planes.

(*iv*) *If yz and zx planes are both planes of symmetry, their line of intersection is an axis of symmetry.*

(*v*) *If axes of x and y are both axes of symmetry, axis of z is also an axis of symmetry.*

In this case there may be no plane of symmetry.

(*vi*) *If all the three coordinate planes are planes of symmetry, then all the coordinate axes are axes of symmetry and the origin is the centre of symmetry.*]

(A) To trace the locus of the equation

$$\frac{\mathbf{x}^2}{\mathbf{a}^2}+\frac{\mathbf{y}^2}{\mathbf{b}^2}+\frac{\mathbf{z}^2}{\mathbf{c}^2}=1. \quad \textbf{(Ellipsoid.)}$$

(1) [**Existence of the centre.**]

If (α, β, γ) be a point on the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1. \qquad \ldots(1)$$

then
$$\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}=1,$$

or
$$\frac{(-\alpha)^2}{a^2}+\frac{(-\beta)^2}{b^2}+\frac{(-\gamma)^2}{c^2}=1.$$

This shows that the point $(-\alpha, -\beta, -\gamma)$ also lies on (1).

The middle point of the line joining these points is (0, 0, 0), the origin.

$\therefore$ (α, β, γ) and $(-\alpha, -\beta, -\gamma)$ are the points on the line through the origin and are equidistant from the origin. The origin bisects all chords passing through it.

$\therefore$ the *origin is the centre of the surface (1)*.

(2) [**Symmetry.**]

If the point (α, β, γ) lies on (1), then the point $(\alpha, \beta, -\gamma)$ also lies on it. The middle point of the line joining these points is $(\alpha, \beta, 0)$. This point lies on the plane $z=0$ and also the line is perpendicular to this plane.

$\therefore$ the coordinate plane XOY bisects all chords which are perpendicular to it.

Similarly YOZ and ZOX planes also bisect all chords perpendicular to them.

Hence (1) is symmetrical with respect to the three coordinate planes* which are called the **principal planes of the ellipsoid.** The three lines of intersection of the three principal planes, taken in pairs, are called the **principal axes**. In this case, the coordinate axes are the principal axes.

(3) [**Axes-intersections.**]

(*i*) The surface (1) meets the x-axis $(y=0, z=0)$ in points $A(a, 0, 0)$ and $A'(-a, 0, 0)$.

(*ii*) The surface (1) meets the y-axis $(z=0, x=0)$ in points $B(0, b, 0)$, $B'(0, -b, 0)$.

(*iii*) The surface (1) meets the z-axis $(x=0, y=0)$ in points $C(0, 0, c)$ and $C'(0, 0, -c)$.

(4) [**Closed surface.**]

If x is numerically greater than a, then from (1), either y^2 or z^2 will be negative, *i.e.*, either y or z will be imaginary.

$\therefore$ x cannot be numerically greater than a,
i.e., the surface (1) lies between two parallel planes $x=a$, $x=-a$.

Similarly, it lies between $y=b$, $y=-b$ and $z=c$, $z=-c$.

$\therefore$ (1) is a closed surface.

(5) [**Section of (1) by planes parallel to the coordinate planes.**]

The section of the surface (1) by the plane $z=k$, which is parallel to the XOY plane, is the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1-\frac{k^2}{c^2},\ z=k \qquad \ldots(A)$$

***Otherwise** $\because$ (1) contains even powers of x, even powers of y, and even powers of z, $\therefore$ (1) is symmetrical about YOZ, ZOX and XOY planes.

Its semi-axes are $a\sqrt{1-\frac{k^2}{c^2}}, b\sqrt{1-\frac{k^2}{c^2}}$.

(*i*) k cannot be numerically greater than c for we have shown that the ellipsoid is a closed surface bounded by the planes

$$z=c, z=-c\ ;\ x=a, x=-a\ ;\ y=b, y=-b.$$

$\therefore$ for values of k greater than c, the ellipse is imaginary.

(*ii*) If $k=c$ or $-c$, the semi-axes of the ellipse are zero.

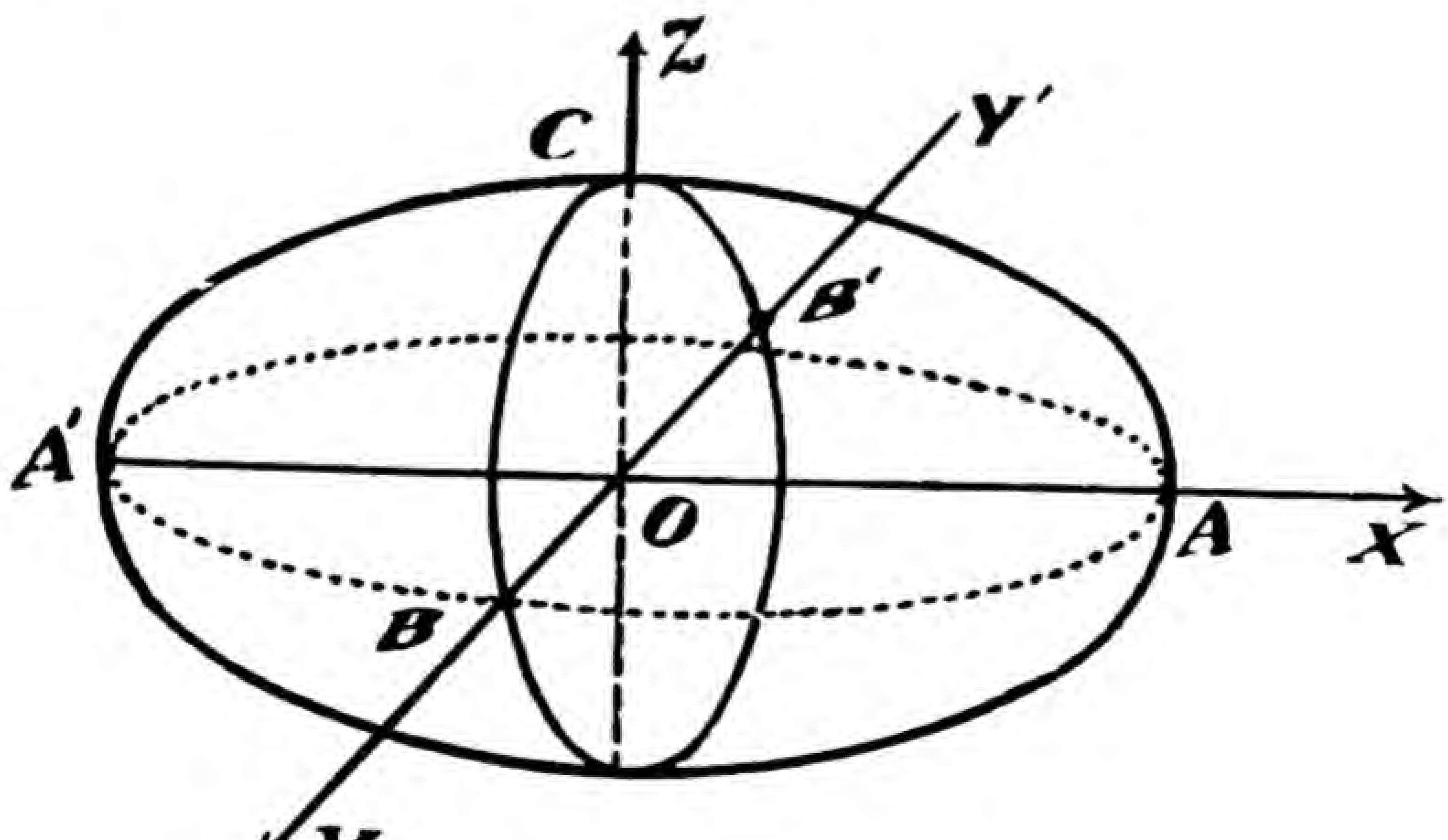
$\therefore$ the section is an infinitely small ellipse. In fact it reduces to the points $(0, 0, c)$ or $(0, 0, -c)$ and the planes $z=c$, or $z=-c$ touch the ellipsoid at these points.

(*iii*) If $k=0$, the section is an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in the XOY plane.

(*iv*) As k varies from $-c$ to c, the ellipse first increases in size and then again diminishes.

The ellipsoid may, therefore be generated by the variable ellipse (A) as k varies from $-c$ to c.

Similarly, it can be shown that the sections by the planes parallel to other coordinate planes are also ellipses and the ellipsoid is generated by them. Hence the shape of the surface is that shown in the figure.



Note 1. *The surface* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^3}=-1$ *is an imaginary ellipsoid.* **(why ?)**

Note 2. *If $b=c$, the section of the surface $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ by the plane parallel to YOZ plane is a circle.*

$\therefore$ *the equation $\frac{x^2}{a^2}+\frac{y^2+z^2}{b^2}=1$ represents an ellipsoid of revolution formed by revolving the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $z=0$, about its major axis. It is called* **prolate spheroid.**

Similarly the equation $\frac{x^2+z^2}{a^2}+\frac{y^2}{b^2}=1$ represents the ellipsoid formed by revolving the same ellipse about the minor axis. It is called **oblate spheroid.**

9·5. Representative equation of a central conicoid.

All the three equations of the three central conicoids given above may be represented by the equation $\mathbf{ax^2+by^2+cz^2=1}$, which is

(*i*) an ellipsoid if a, b, c are all positive,

(*ii*) a hyperboloid of one sheet if two are positive and one negative,

(*iii*) a hyperboloid of two sheets if two are negative and one positive,

and (*iv*) a virtual ellipsoid if all are negative.

Note 1. *The conicoid represented by the equation $ax^2+by^2+cz^2=1$ has origin as the centre.*

Note 2. *The above equation is called the standard equation of the central conicoid.*

9·6. Diameter : Def.

The diameter of a conicoid is the chord of the conicoid which passes through the centre of the conicoid.

9·7. Diametral plane : Def.

The plane which bisects a system of parallel chords of a conicoid is called a diametral plane. Thus the plane YOZ is a diametral plane of OX, since it bisects all chords parallel to OX.

9·8. Conjugate diametral planes : Def.

If three diametral planes be such that each bisects chords parallel to the line of intersection of the other two, they are said to be conjugate diametral planes. Thus the coordinate planes are conjugate diametral planes of the central conicoid

$$ax^2+by^2+cz^2=1.$$

9·9. Conjugate diameters : Def.

Three diameters, which are such that a plane through any two bisects chords parallel to the third, are called conjugate diameters of the central conicoid. Thus, the coordinate axes are conjugate diameters of $ax^2+by^2+cz^2=1$.

9·10. Principal planes : Def.

The diametral planes, which are at right angles to the chords which they bisect, are called principal planes.

9·11. Principal Axes : Def.

The lines of the intersection of principal planes, taken two by two, are called principal axes.

9·12. Note. Important.

If axes are rectangular, $\mathbf{ax^2+by^2+cz^2=1}$ represents a central conicoid referred to its principal axes as axes of coordinates.

SECTION II

9·13. To find the points of intersection of the line

$$\frac{\mathbf{x}-\alpha}{\mathbf{l}}=\frac{\mathbf{y}-\beta}{\mathbf{m}}=\frac{\mathbf{z}-\gamma}{\mathbf{n}}$$

and the central conicoid $\mathbf{ax^2+by^2+cz^2=1}$.

Let the given line and given conicoid be

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r, \text{ say} \qquad \ldots(1)$$

and
$$ax^2+by^2+cz^2=1 \qquad \ldots(2)$$
respectively.

Any point on (1) is $(\alpha+lr, \beta+mr, \gamma+nr)$.

It if lies on (2), then

$$a(\alpha+lr)^2+b(\beta+mr)^2+c(\gamma+nr)^2=1,$$

or,
$$r^2(al^2+bm^2+cn^2)+2r(al\alpha+bm\beta+cn\gamma)$$
$$+(a\alpha^2+b\beta^2+c\gamma^2-1)=0 \qquad \ldots(3)$$

This is a quadratic equation in r.

$\therefore$ it has two roots, say r_1 and r_2.

Hence the two points of intersection of (1) and (2) are

$$(\alpha+lr_1,\ \beta+mr_1,\ \gamma+nr_1)$$

and
$$(\alpha+lr_2,\ \beta+mr_2,\ \gamma+nr_2).$$

Note 1. If l, m, n be **actual** direction cosines and, P be the point (α, β, γ) and Q and R be the points of intersection of (1) and (2), then r_1 and r_2 give the lengths of PQ and PR.

Note 2. *Any plane section of a conicoid is a conic for every line in its plane meets the curve of intersection of the central conicoid and the plane in two points only.*

9·14. Tangent Plane at a point.

To find the equation of the tangent plane at the point (x_1, y_1, z_1) of the central conicoid $ax^2+by^2+cz^2=1$.

The equation of the given conicoid is

$$ax^2+by^2+cz^2=1 \qquad \text{...(1)}$$

Equations of **any** line through (x_1, y_1, z_1) is

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r, \text{ say.} \qquad \text{...(2)}$$

Any point on this line is (x_1+lr, y_1+mr, z_1+nr).

If it lies on (1), then

$$a(x_1+lr)^2+b(y_1+mr)^2+c(z_1+nr)^2=1,$$

or,
$$r^2(al^2+bm^2+cn^2)+2r(alx_1+bmy_1+cnz_1)$$
$$+(ax_1^2+by_1^2+cz_1^2-1)=0 \qquad \text{...(3)}$$

$\because$ (x_1, y_1, z_1) lies on (1),

$\therefore$ $ax_1^2+by_1^2+cz_1^2=1$ ...(4)

$\therefore$ (3) becomes $r^2(al^2+bm^2+cn^2)+2r(alx_1+bmy_1+cnz_1)=0.$

$\therefore$ its one root is zero.

The line (2) will be a tangent line to (1) if the other root is also zero.

This requires that $alx_1+bmy_1+cnz_1=0$ ...(5)

Eliminating l, m, n between (2) and (5), the locus of all tangent lines is
$$ax_1(x-x_1)+by_1(y-y_1)+cz_1(z-z_1)=0,$$

or,
$$axx_1+byy_1+czz_1=ax_1^2+by_1^2+cz_1^2,$$

Hence $\mathbf{axx_1+byy_1+czz_1=1}$, using (4), which is the required equation of the tangent plane at (x_1, y_1, z_1).

Note. *The equation of the tangent plane at any point (α, β, γ) of the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \text{ is}$$

$$\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}+\frac{\gamma z}{c^2}=1. \qquad \textit{(Delhi Hons., 1959)}$$

The students are advised to verify this result.

9·15. Condition of tangency.

To find the condition that the plane $lx+my+nz=p$ should touch the conicoid $ax^2+by^2+cz^2=1$.

The equations of the given conicoid and given plane are respectively

$$ax^2+by^2+cz^2=1 \qquad ...(1)$$

and

$$lx+my+nz=p. \qquad ...(2)$$

Let (2) touch (1) at the point (α, β, γ).

The equation of the tangent plane at (α, β, γ) is

$$ax\alpha+by\beta+cz\gamma=1. \qquad ...(3)$$

$\because$ (2) and (3) represent the same plane,

$\therefore$ we have on comparing coefficients of like terms,

$$\frac{a\alpha}{l}=\frac{b\beta}{m}=\frac{c\gamma}{n}=\frac{1}{p}.$$

$\therefore$
$$\alpha=\frac{l}{ap},\ \beta=\frac{m}{bp},\ \gamma=\frac{n}{cp} \qquad ...(4)$$

$\because$ (α, β, γ) lies on (1),

$\therefore$
$$a\alpha^2+b\beta^2+c\gamma^2=1. \qquad ...(5)$$

From (4) and (5), we have

$$a\left(\frac{l^2}{a^2p^2}\right)+b\left(\frac{m^2}{b^2p^2}\right)+c\left(\frac{n^2}{c^2p^2}\right)=1,$$

or,
$$\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2,$$

which is the required condition.

Note 1. Point of contact.

From (4), the point of contact is

$$\left(\frac{l}{ap},\ \frac{m}{bp},\ \frac{n}{cp}\right).$$

Cor. To find the equations of the two tangent planes to the central conicoid $ax^2+by^2+cz^2=1$, which are parallel to the plane $lx+my+nz=0$.

The equation of plane parallel to the plane

$$lx+my+nz=0 \text{ is } lx+my+nz=p. \qquad ...(1)$$

If it touches the given conicoid, then

$$\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2,$$

or,
$$p=\pm\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}}.$$

Substituting these values of p in (1), we have

$$\mathbf{lx+my+nz=\pm\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}}},$$

which are the equations of the tangent planes to the conicoid for all values of l, m, n.

Note 2. Case of ellipsoid.

The plane $lx+my+nz=p$ touches the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \text{ if } \mathbf{a^2l^2+b^2m^2+c^2n^2=p^2}.$$

The point of contact in this case is

$$\left(\frac{\mathbf{a^2l}}{\mathbf{p}}, \frac{\mathbf{b^2m}}{\mathbf{p}}, \frac{\mathbf{c^2n}}{\mathbf{p}}\right).$$

Ex. *Obtain the tangent planes to the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

which are parallel to the plane

$$lx+my+nz=0.$$

(Delhi Hons., 1959 ; Punjab Hons., 1961 ; Baroda B.Sc., 1960)

[**Ans.** $lx+my+nz=\pm\sqrt{a^2l^2+b^2m^2+c^2n^2}$.]

EXAMPLES IX (A)

Type I. Ex. 1. Any three mutually orthogonal lines drawn through a fixed point C meet the conicoid $ax^2+by^2+cz^2=1$ in P_1, P_2; Q_1, Q_2 ; R_1, R_2 respectively.

Prove that $$\frac{1}{CP_1.CP_2}+\frac{1}{CQ_1.CQ_2}+\frac{1}{CR_1.CR_2}$$

and $$\frac{P_1P_2^2}{CP_1^2.CP_2^2}+\frac{Q_2Q_2^2}{CQ_1^2.CQ_2^2}+\frac{R_1R_2^2}{CR_1^2.CR_2^2}$$

are constants. *(Osmania, 1961, second part)*

Sol. Let the fixed point C be (x_1, y_1, z_1)

Let the equations of CP_1P_2 be

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n}=r,$$

where l_1, m_1, n_1 are **actual** direction cosines.

Any point on this line is $(x_1+l_1r, y_1+m_1r, z_1+n_1r)$.

If it lies on the given conicoid, then

$$r^2(al_1^2+bm_1^2+cn_1^2)+2r(ax_1l_1+by_1m_1+cz_1n_1)+(ax_1^2+by_1^2+cz_1^2-1)=0 \quad \ldots(1)$$

Let r_1 and r_2 be its roots. These two roots correspond to the measures of the distances CP_1 and CP_2.

Then $\qquad CP_1=r_1,\ CP_2=r_2.$

$\therefore \qquad P_1P_2=r_1\sim r_2.$

Let l_2, m_2, n_2 and l_3, m_3, n_3 be the actual direction cosines of the lines CQ_1Q_2 and CR_1R_2 respectively.

$\because$ the lines CP_1P_2, CQ_1Q_2, CR_1R_2 are orthogonal,

$\therefore \quad \Sigma m_1n_1=\Sigma n_1l_1=\Sigma l_1m_1=0$

and $\quad \Sigma l_1^2=1=\Sigma m_1^2=\Sigma n_1^2. \quad \ldots(2)$

Now, $\displaystyle \frac{1}{CP_1.CP_2}=\frac{1}{r_1r_2}=\frac{al_1^2+bm_1^2+cn_1^2}{ax_1^2+by_1^2+cz_1^2-1}$,

using (1)

Similarly, $\displaystyle \frac{1}{CQ_1.CQ_2}=\frac{al_2^2+bm_2^2+cn_2^2}{ax_1^2+by_1^2+cz_1^2-1}$

and $\displaystyle \frac{1}{CR_1.CR_2}=\frac{al_3^2+bm_3^2+cn_3^2}{ax_1^2+by_1^2+cz_1^2-1}$

$\displaystyle \therefore \quad \frac{1}{CP_1.CP_2}+\frac{1}{CQ_1.CQ_2}+\frac{1}{CR_1.CR_2}=\frac{a+b+c}{ax_1^2+by_1^2+cz_1^2-1}$,

using (2), which is constant.

$$\text{Again, } \frac{P_1P_2^2}{CP_1^2.CP_2^2}=\frac{(r_1\sim r_2)^2}{r_1^2r_2^2}=\frac{(r_1+r_2)^2-4r_1r_2}{r_1^2r_2^2}$$

$$=\frac{(r_1+r_2)^2}{r_1^2r_2^2}-\frac{4}{r_1r_2}$$

$$=\frac{4[ax_1l_1+by_1m_1+cz_1n_1]^2}{(ax_1^2+by_1^2+cz_1^2-1)^2}-\frac{4(al_1^2+bm_1^2+cn_1^2)}{ax_1^2+by_1^2+cz_1^2-1}$$

$$\text{Similarly, } \frac{Q_1Q_2^2}{CQ_1^2.CQ_2^2}=\frac{4[ax_1l_2+by_1m_2+cz_1n_2]^2}{(ax_1^2+by_1^2+cz_1^2-1)^2}-\frac{4(al_2^2+bm_2^2+cn_2^2)}{ax_1^2+by_1^2+cz_1^2-1}$$

$$\text{and } \frac{R_1R_2^2}{CR_1^2.CR_2^2}=\frac{4[ax_1l_3+by_1m_3+cz_1n_3]^2}{(ax_1^2+by_1^2+cz_1^2-1)^2}-\frac{4(al_3^2+bm_3^2+cn_3^2)}{(ax_1^2+by_1^2+cz_1^2-1)}$$

$$\therefore \quad \frac{P_1P_2^2}{CP_1^2\cdot CR_2^2}+\frac{Q_1Q_1^2}{CQ_1^2\cdot CQ_2^2}+\frac{R_1R_2^2}{CR_1^2\cdot CR_2^2}$$

$$=\frac{4(a^2x_1^2+b^2y_1^2+c^2z_1^2)}{(ax_1^2+by_1^2+cz_1^2-1)^2}-\frac{4(a+b+c)}{ax_1^2+by_1^2+cz_1^2-1}$$

which is constant.

Ex. 2. A is a given point and POP′ any diameter of a central conicoid. If OQ and OQ′ are the semi-diameters parallel to AP and AP′, prove that $\dfrac{AP^2}{OQ^2}+\dfrac{AP'^2}{OQ'^2}$ is constant.

Ex. 3. A line through a given point A meets a central conicoid in P and Q. If OD is the diameter parallel to APQ, prove that AP.AQ : OD² is constant.

Type II. Ex. 1. If P be the point of contact of a tangent plane ABC to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

and PD, PE, PF are perpendiculars from P on the axis, prove that OD.OA=a², OE.OB=b², OF.OC=c², A, B, C being the points where the tangent plane at P meets the coordinate axes.

Sol. Let P be the point (x_1, y_1, z_1).

Equation of the tangent plane at (x_1, y_1, z) is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=1 \qquad ...(1)$$

$\because$ (1) cuts the axes at A, B, C respectively,

$$\therefore \qquad OA=\frac{a^2}{x_1}, \quad OB=\frac{b^2}{y_1} \text{ and } OC=\frac{c^2}{z_1}.$$

$\because$ PD, PE and PF are perpendiculars from P on the axes,

$$\therefore \qquad OD=x_1, \quad OE=y_1 \text{ and } OF=z_1.$$

Now, $OD.OA=x_1.\dfrac{a^2}{x_1}=a^2, \quad OE.OB=y_1.\dfrac{b^2}{y_1}=b^2$

and $\qquad OF.OC=z_1.\dfrac{c^2}{z_1}=c^2.$

Ex. 2. Find the equation of the tangent plane to the surface $3x^2+y^2+z^2=23$ at the point (2, 3, 0). *(Raj. Engg., 1956)*

[Ans. $2x+y=7.$**]**

Type III. Ex. 1. Prove that the equation of the two tangent planes to the conicoid

$$ax^2+by^2+cz^2=1$$

which pass through the line

$$u\equiv lx+my+nz-p=0,$$
$$u'\equiv l'x+m'y+n'z-p'=0,$$

is

$$u^2\left(\frac{l'^2}{a}+\frac{m'^2}{b}+\frac{n'^2}{c}-p'^2\right)-2uu'\left(\frac{ll'}{a}+\frac{mm'}{b}+\frac{nn'}{c}-pp'\right)$$
$$+u'^2\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}-p^2\right)=0.$$ *(Punjab Hons., 1959)*

Sol. The equation of **any** plane passing through the given line is

$$u+\lambda u'=0 \qquad ...(1)$$

It will touch the given conicoid if

$$\frac{(l+\lambda l')^2}{a}+\frac{(m+\lambda m')^2}{b}+\frac{(n+\lambda n')^2}{c}=(p+\lambda p')^2 \quad ...(2) \quad \text{(Art. 9·15)}$$

This is a quadratic equation in λ.

$\therefore$ it has two roots.

Hence there are two tangent planes passing through the given line.

Eliminating λ between (1) and (2), we have

$$\frac{(u'l-ul')^2}{a}+\frac{(u'm-um')^2}{b}+\frac{(u'n-un')^2}{c}=(u'p-up')^2,$$

or, $$u^2\left(\frac{l'^2}{a}+\frac{m'^2}{b}+\frac{n'^2}{c}-p'^2\right)-2uu'\left(\frac{ll'}{a}+\frac{mm'}{b}+\frac{nn'}{c}-pp'\right)$$

$+u'^2\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}-p^2\right)=0$, which are the equations of the required two tangent planes.

Ex. 2. (*i*) Find the to equations two tangent planes to $2x^2-6y^2+3z^2=5$, which pass through the line $x+9y-3z=0$, $3x-3y+6z-5=0$.

(*Punjab Hons., 1946*)

[**Ans.** $4x+6y+3z=5$, $2x-12y+9z=5$.]

(*ii*) Find equations of two planes that can be drawn through the line $x=4$, $3y+4z=0$ to touch the conicoid $x^2+3y^2-6z^2=4$. (*Delhi Hons., 1953*)

[**Ans.** $x+9y+12z=4$, $x-9y-12z=4$]

Type IV. Ex. 1. Director sphere. Find the locus of the point of intersection of three mutually perpendicular tangent planes to the central conicoid $ax^2+by^2+cz^2=1$. (*Karnatak, 1961*)

Sol. The equation of the given central conicoid is

$$ax^2+by^2+cz^2=1 \quad ...(1)$$

Let the three mutually perpendicular tangent planes to (1) be

$$l_1x+m_1y+n_1z=\sqrt{\frac{l_1^2}{a}+\frac{m_1^2}{b}+\frac{n_1^2}{c}} \quad ...(2)$$

$$l_2x+m_2y+n_2z=\sqrt{\frac{l_2^2}{a}+\frac{m_2^2}{b}+\frac{n_2^2}{c}} \quad ...(\ \ )$$

and $$l_3x+m_3y+n_3z=\sqrt{\frac{l_3^2}{a}+\frac{m_3^2}{b}+\frac{n_3^2}{c}} \quad ...(3)$$

$\because$ l_1, m_1, n_1 ; $l_2, m_2, n_2, l_3, m_3, n_3$ are direction cosines of three mutually perpendicular lines, being the normals to the three mutually perpendicular tangent planes (2), (3) and (4),

$\therefore$ $l_1^2+l_2^2+l_3^2=1$, etc., $l_1l_2+m_1m_2+n_1n_2=0$, etc. ; $l_1m_1+l_2m_2+l_3m_3=0$, etc. ; $l_1^2+m_1^2+n_1^2=1$, etc.

The locus of the point of intersection of (2), (3) and (4) is obtained by eliminating l_1, m_1, n_1, ; l_2, m_2, n_2 and l_3, m_3, n_3 between them.

Squaring (2), (3) and (4) and then adding, we have

$$x^2(l_1^2+l_2^2+l_3^2)+y^2(m_1^2+m_2^2+m_3^2)+z^2(n_1^2+n_2^2+n_3^2)+2yz(m_1n_1+m_2n_2+m_3n_3)$$
$$+2zx(n_1l_1+n_2l_2+n_3l_3)+2xy(l_1m_1+l_2m_2+l_3m_3)$$
$$=\frac{1}{a}(l_1^2+l_2^2+l_3^2)+\frac{1}{b}(m_1^2+m_2^2+m_3^2)+\frac{1}{c}(n_1^2+n_2^2+n_3^2),$$

or, $$x^2+y^2+z^2=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

Note. *This is the equation of the* **'Director sphere'** *of the conicoid.*

Ex. 2. Prove that the locus of points from which three mutually perpendicular planes can be drawn to touch the ellipse

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $z=0$ is the sphere $x^2+y^2+z^2=a^2+b^2$.

(Punjab, (Pakistan), 1956S ; Sagar B.Sc., 1962)

Ex. 3. Prove that the locus of the foot of the perpendicular drawn from the centre of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ to any of its tangent planes is $a^2x^2+b^2y^2+c^2z^2=(x^2+y^2+z^2)^2$.

Type V. Ex. 1. If 2r is the distance between two parallel tangent planes to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, prove that the line through the origin perpendicular to the planes lie on the cone

$$x^2(a^2-r^2)+y^2(b^2-r^2)+z^2(c^2-r^2)=0.$$

(Punjab Hons., 1961 ; Delhi Hons., 1959)

Sol. The parallel tangent planes to the given ellipsoid are

$$lx+my+nz=\pm\sqrt{a^2l^2+b^2m^2+c^2n^2} \qquad ...(1)$$

By the question,

$$\frac{2\sqrt{a^2l^2+b^2m^2+c^2n^2}}{l^2+m^2+n^2}=2r,$$

or $$(a^2-r^2)l^2+(b^2-r^2)m^2+(c^2-r^2)n^2=0 \qquad ...(2)$$

A line through the origin and perpendicular to (1) is

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \qquad ...(3)$$

∴ from (2), the line (3) generates the cone

$$(a^2-r^2)x^2+(b^2-r^2)y^2+(c^2-r^2)z^2=0.$$

Ex. 2. If the line of intersection of two perpendicular tangent planes to the ellipsoid whose equation referred to rectangular axes is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{a^2}=1$ passes through the fixed point (0, 0, k), show that it lies on the cone

$$x^5(b^2+c^2-k^2)+y^2(c^2+a^2-k^2)+(z-k)^2(a^2+b^2)=0.$$

(Delhi Hons., 1960 ; Karnatak, 1961)

Ex. 3. Tangent planes are drawn to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ through the point (α, β, γ). Prove that the perpendiculars to them from the origin generate the cone $a^2x^2+b^2y^2+c^2z^2=(\alpha x+\beta y+\gamma z)^2$.

(*Marathwada T.D.C., 1961 ; Sagar B.Sc., 1952*)

SECTION III

9·16. Polar plane of a point : Def.

If any secant APQ, through a given point A, meets a conicoid in P and Q, then the locus of R, the harmonic conjugate of A with respect to P and Q (*i.e.*, AP, AQ and AR are in Harmonic progression), is called the polar plane of A with respect to the conicoid.

9·17. To find the polar plane of a given point $A(x_1, y_1, z_1)$ with respect to the conicoid $ax^2+by^2+cz^2=1$.

The given conicoid is $\quad ax^2+by^2+cz^2=1 \quad$ ...(1)

Let the equations of **any** secant APQ through the point

$$A(x_1, y_1, z_1) \text{ be } \frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r, \quad \text{...(2)}$$

say, where l, m, n are the actual direction cosines.

Let r_1 and r_2 be the measures of AP and AQ.

$\therefore$ From Art. 9·13, r_1 and r_2 are the roots of the quadratic equation

$$r^2(al^2+bm^2+cn^2)+2r(ax_1l+by_1m+cz_1n) +(ax_1^2+by_1^2+cz_1^2-1)=0 \quad \text{...(3)}$$

Let ρ be the measure of AR.

$$\therefore \quad AR=\rho=\frac{2r_1r_2}{r_1+r_2}.$$

because AR is the harmonic mean between AP and AQ,

or $$AR=\rho=\frac{ax_1^2+by_1^2+cz_1^2-1}{-(alx_1+bmy_1+cnz_1)} \quad \text{...(4)}$$

using (3).

Now, if R be the point (λ, μ, ν) which lies on the line (2) for $r=\rho$, we have

$$\lambda-x_1=l\rho,\ \mu-y_1=m\rho$$

and $$\nu-z_1=n\rho \quad \text{...(5)}$$

where ρ is the distance of R from A.

Eliminating ρ between (4) and (5), we have

$$ax_1(\lambda-x_1)+by_1(\mu-y_1)+cz_1(\nu-z_1)=-(ax_1^2+by_1^2+cz_1^2-1),$$

or
$$ax_1\lambda+by_1\mu+cz_1\nu=0.$$

$\therefore$ locus of $R(\lambda, \mu, \nu)$ is the plane

$$\mathbf{axx_1+byy_1+czz_1=1},$$

which is the required polar plane of the point (x_1, y_1, z_1) with respect to the conicoid (1).

Note 1. *The equation of the polar plane is of the* **same** *form as that of the tangent plane at the given point.*

Note 2. *The point A is called the pole of its polar plane.*

9·18. Conjugate points : Def.

Two points, such that the polar plane of either with respect to a conicoid passes through the other, are called **conjugate points.**

9·19. Conjugate planes : Def.

Two planes, such that the pole of either with respect to a conicoid lie on the other, are called **conjugate planes.**

9·20. Polar lines with respect to a conicoid : Def.

Two lines, which are such that the polar plane of any point on either with respect to a conicoid passes through the other, are called the **polar lines** with respect to the conicoid.

9·21. To find the equations to the polar line of the line

$$\frac{\mathbf{x-x_1}}{\mathbf{l}}=\frac{\mathbf{y-y_1}}{\mathbf{m}}=\frac{\mathbf{z-z_1}}{\mathbf{n}}$$

with respect to the conicoid $\mathbf{ax^2+by^2+cz^2=1}$.

The equations of the given line are

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r, \text{ say} \qquad \ldots(1)$$

and the equation of the given conicoid is

$$ax^2+by^2+cz^2=1 \qquad \ldots(2)$$

Any point on (1) is (x_1+lr, y_1+mr, z_1+nr).

The polar plane of this point with respect to (2) is

$$ax(x_1+lr)+by(y_1+mr)+cz(z_1+nr)=1, \qquad \text{(Art. 9·17)}$$

or,
$$(axx_1+byy_1+czz_1-1)+r(alx+bmy+cnz)=0.$$

This plane, for all values of r, passes through the line given by the intersection of the planes

$$axx_1+byy_1+czz_1=1$$

and $$alx+bmy+cnz=0.$$

Hence the equations of the polar line of (i) are

$$\mathbf{axx_1+byy_1+czz_1=1}$$

and $$\mathbf{alx+bmy+cnz=0}.$$

Ex. *Obtain the equations of the polar of a given line with respect to the ellipsoid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. *(Karnatak M.Sc., 1961)*

EXAMPLES IX (B)

Type I. Ex. 1. Prove that the locus of the pole of the plane

$$\mathbf{lx+my+nz=p}$$

with respect to the system of conicoid $\frac{x^2}{a^2+k}+\frac{y^2}{b^2+k}+\frac{z^2}{c^2+k}=1$, where k is a parameter is the straight line perpendicular to the given plane.

Sol. Let (α, β, γ) be the pole.

Its polar with respect to the given conicoid is

$$\frac{x\alpha}{a^2+k}+\frac{y\beta}{b^2+k}+\frac{z\gamma}{c^2+k}=1 \qquad ...(1)$$

Now (1) is identical with $\qquad lx+my+nz=p \qquad ...(2)$

$\therefore$ Comparing coefficients in (1) and (2), we have

$$\frac{l(a^2+k)}{\alpha}=\frac{m(b^2+k)}{\beta}=\frac{n(c^2+k)}{\gamma}=\frac{p}{1}$$

$$\therefore \quad a^2+k=\frac{p\alpha}{l},\ b^2+k=\frac{\beta p}{m},\ c^2+k=\frac{p\gamma}{n},$$

or, $$a^2-\frac{p\alpha}{l}=b^2-\frac{\beta p}{m}=c^2-\frac{p\gamma}{n}=-k$$

or $$\frac{\frac{l}{p}a^2-\alpha}{l/p}=\frac{\frac{m}{p}b^2-\beta}{m/p}=\frac{\frac{n}{p}c^2-\gamma}{n/p},$$

or, $$\frac{\alpha-\frac{la^2}{p}}{l}=\frac{\beta-\frac{mb^2}{p}}{m}=\frac{\gamma-\frac{nc^2}{p}}{n}.$$

$\therefore$ locus of (α, β, γ) is the straight line.

$$\frac{x-\frac{la^2}{p}}{l}=\frac{y-\frac{mb^2}{p}}{m}=\frac{z-\frac{nc^2}{p}}{n}.$$

This straight line is clearly perpendicular to the plane

$$lx+my+nz=p.$$

Ex. 2. Prove that the locus of the poles of the tangent planes of $ax^2+by^2+cz^2=1$ with respect to $\alpha x^2+\beta y^2+\gamma z^2=1$ is the conicoid

$$\frac{\alpha^2x^2}{a}+\frac{\beta^2y^2}{b}+\frac{\gamma^2z^2}{c}=1.$$

Ex. 3. Reciprocal property. If the polar plane of a point P with respect to a central conicoid passes through a point Q, prove that the polar plane of Q passes through P.

Type II. Ex. 1. Find the equations of the polar line of

$$\frac{\mathbf{x}+1}{2}=\frac{\mathbf{y}-2}{3}=\mathbf{z}+3$$

with respect to the sphere $\mathbf{x}^2+\mathbf{y}^2+\mathbf{z}^2=1$. *(Punjab, 1956)*

Sol. Here $x_1=-1,\quad y_1=2,\quad z_1=-3,\quad l=2,\quad m=3,\quad n=1,$
$a=1,\quad b=1,\quad c=1.$

From Art. 9·21, the required equations of the polar line of the given line are $\quad -x+2y-3z=1\quad$ and $\quad 2x+3y+z=0,$

or, $\quad x-2y+3z+1=0$ and $\quad 2x+3y+z=0.$

Ex. 2. Prove that the polar line of

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$$

with respect to the conicoid $x^2-2y^2+3z^2-4=0$ is

$$\frac{x+6}{3}=\frac{y-2}{3}=\frac{z-2}{1}.$$

Type III. Ex. 1. Find the conditions that the lines

$$\frac{\mathbf{x}-\alpha}{\mathbf{l}}=\frac{\mathbf{y}-\beta}{\mathbf{m}}=\frac{\mathbf{z}-\gamma}{\mathbf{n}}\quad\textbf{and}\quad\frac{\mathbf{x}-\alpha'}{\mathbf{l}'}=\frac{\mathbf{y}-\beta'}{\mathbf{m}'}=\frac{\mathbf{z}-\gamma'}{\mathbf{n}'}$$

must be polar lines with respect to the ellipsoid

$$\frac{\mathbf{x}^2}{\mathbf{a}^2}+\frac{\mathbf{y}^2}{\mathbf{b}^2}+\frac{\mathbf{z}^2}{\mathbf{c}^2}=1.\qquad \textit{(Karnatak M.Sc., 1961)}$$

Sol. Let PQ and RS be the given lines

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}\qquad ...(1)$$

and

$$\frac{x-\alpha'}{l'}=\frac{y-\beta'}{m'}=\frac{z-\gamma'}{n'}\qquad ...(2)$$

The equations of the polar line of PQ are

$$\left.\begin{aligned}\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}+\frac{\gamma z}{c^2}&=1\\ \frac{lx}{a^2}+\frac{my}{b^2}+\frac{nz}{c^2}&=0\end{aligned}\right\}\qquad ...(2)$$

Now, if the polar line of PQ is the given line RS, then RS lies on both the planes of (2). The conditions for this are that

(*i*) the point $(\alpha', \beta', \gamma')$ on the line RS must satisfy both the equations of (2) and

(*ii*) the line RS must be perpendicular to the normals of the planes of (2).

$\therefore$ the required conditions are

$$\frac{\alpha\alpha'}{a^2}+\frac{\beta\beta'}{b^2}+\frac{\gamma\gamma'}{c^2}=1,\quad \frac{\alpha l'}{a^2}+\frac{\beta m'}{b^2}+\frac{\gamma n'}{c^2}=0,$$

$$\frac{l\alpha'}{a^2}+\frac{m\beta'}{b^2}+\frac{m\gamma'}{c^2}=0,\quad \frac{ll'}{a^2}+\frac{mm'}{b^2}+\frac{nn'}{c^2}=0.$$

Ex. 2. Find the condition that the line

$\dfrac{x-\alpha}{l}=\dfrac{y-\beta}{m}=\dfrac{z-\gamma}{n}$ should intersect the polar of the line

$\dfrac{x-\alpha'}{l'}=\dfrac{y-\beta'}{m'}=\dfrac{z-\gamma'}{n'}$ with respect to the conicoid

$ax^2+by^2+cz^2=1.$

[**Ans.** $(a\alpha l'+b\beta m'+c\gamma n')(a\alpha' l+b\beta' m+c\gamma' n)$
$=(all'+bmm'+cnn')(a\alpha\alpha'+b\beta\beta'+c\gamma\gamma'-1)$].

Type IV. Ex. 1. Prove that if AB intersects the polar of PQ, then PQ intersects the polar of AB. (*Karnatak M.Sc., 1961*)

Sol. Let RS be the polar of PQ intersecting AB in C, and EF be the polar of AB.

Let us consider the polar plane of C.

$\because$ RS, PQ are polar lines,

$\therefore$ the polar plane of C (a point on RS) passes through PQ.

Also, $\because$ AB and EF are polar lines,

$\therefore$ the polar plane of C (a point on AB), passes through EF.

$\therefore$ the polar plane of C passes through PQ, EF,
i.e., PQ and EF are coplanar.

$\therefore$ EF, the polar of AB, intersects PQ,

i.e., PQ intersects the polar of AB.

Aliter. Let AB and PQ be the lines

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad \ldots(1)$$

and

$$\frac{x-\alpha'}{l'}=\frac{y-\beta'}{m'}=\frac{z-\gamma'}{n'} \qquad \ldots(2)$$

respectively.

The equations of the polar line of the line (2) with respect to the conicoid $ax^2+by^2+cz^2=1$ is the line EF given by

$$\left.\begin{aligned} a\alpha'x+b\beta'y+c\gamma'z-1&=0 \\ \text{and}\quad al'x+bm'y+cn'y\quad &=0 \end{aligned}\right\} \qquad ...(3)$$

If the line AB intersects EF given by (3), then any point $(\alpha+lr,\ \beta+mr,\ \gamma+nr)$ on AB must satisfy (3), for some value of r.

$$\therefore\quad a\alpha'(\alpha+lr)+b\beta'(\beta+mr)+c\gamma'\ (\gamma+nr)=1$$

and $$al'(\alpha+lr)+bm'(\beta+mr)+cn'(\gamma+nr)=0,$$

or, $$(a\alpha\alpha'+b\beta\beta'+c\gamma\gamma'-1)+r(a\alpha'l+b\beta'm+c\gamma'n)=0$$

and $$(a\alpha l'+b\beta m'+c\gamma n')+r(all'+bmm'+cnn')=0$$

Eliminating r between these equations, we obtain the condition that must hold in order that AB may intersect EF.

$$\begin{aligned}\therefore\quad &(a\alpha\alpha'+b\beta\beta'+c\gamma\gamma'-1)(all'+bmm'+cnn') \\ &=(a\alpha'l+b\beta'm+c\gamma'n)(a\alpha l'+b\beta m'+c\gamma n')\end{aligned} \qquad ...(4)$$

The symmetry of the result (4) shows that if the line AB intersects the polar of PQ, then the line PQ will intersect the polar of AB.

Ex. 2. Show that the polar of a given line with respect to a central conicoid is the chord of contact of the two tangent planes through the line.

Ex. 3. Show that the polar of the line joining two given points is the line of intersection of their polar lines with respect to the conicoid.

SECTION IV

9·22. Section with a given centre.

To find the locus of chords of the central conicoid $ax^2+by^2+cz^2=1$ which are bisected at the point (x_1, y_1, z_1).

The equation of the conicoid is $ax^2+by^2+cz^2=1$ ...(1)

Equations of **any** chord of (1) passing through $(x_1,\ y_1,\ z_1)$ are

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r, \text{ say} \qquad ...(2)$$

Any point on this line is $(x_1+lr,\ y_1+mr,\ z_1+nr)$. If it lies on (1), then

$$a(x_1+lr)^2+b(y_1+mr)^2+c(z_1+nr)^2=1,$$

or, $$\begin{aligned} &r^2(al^2+bm^2+cn^2)+2r(alx_1+bmy_1+cnz_1) \\ &\quad+(ax_1^2+by_1^2+cz_1^2-1)=0 \end{aligned} \qquad ...(3)$$

This is a quadratic equation in r and therefore has two roots,

$\because\quad (x_1,\ y_1,\ z_1)$ is the middle point of the chord (2)

$\therefore$ (3) has roots equal in magnitude and opposite in signs, *i.e.*, the sum of the roots of (3) is zero.

$$\therefore \qquad alx_1+bmy_1+cnz_1=0 \qquad \ldots(4)$$

The required locus is obtained by eliminating l, m, n between (2) and (4).

$$\therefore \qquad a(x-x_1)x_1+b(y-y_1)y_1+c(z-z_1)z_1=0,$$

or, $$\mathbf{axx_1+byy_1+czz_1=ax_1^2+by_1^2+cz_1^2},$$

which is the required equation.

Note 1. *If $S\equiv ax^2+by^2+cz^2-1=0$ be the equation of the conicoid and $S_1\equiv ax_1^2+by_1^2+cz_1^2-1$ and $T\equiv axx_1+byy_1+czz_1-1=0$ be the equation of the tangent plane at (x_1, y_1, z_1), then the above equation can be written as*

$$\mathbf{T=S_1}.$$

Note 2. *The section of the conicoid by this plane is a conic such that all chords passing through (x_1, y_1, z_1) are bisected at it, i.e., (x_1, y_1, z_1) is the centre of the conic. This equation determines the equation of the section of the conicoid whose centre is the given point.*

9·23. To find the locus of the middle points of a system of chords of the conicoid $\mathbf{ax^2+by^2+cz^2=1}$ which are parallel to the line $\mathbf{\dfrac{x}{l}=\dfrac{y}{m}=\dfrac{z}{n}}$.

Let (λ, μ, ν) be the middle point of one chord of the system of chords of the conicoid $ax^2+by^2+cz^2=1$ ...(1), which are parallel to the line $\dfrac{x}{l}=\dfrac{y}{m}=\dfrac{z}{n}$.

$\therefore$ equations of this chord are

$$\frac{x-\lambda}{l}=\frac{y-\mu}{m}=\frac{z-\nu}{n} \qquad \ldots(2)$$

If (λ, μ, ν) be the middle point of the chord (2),

then $$al\lambda+bm\mu+cn\nu=0 \qquad \ldots(3) \qquad \text{(Art. 9·22)}$$

$\therefore$ locus of (λ, μ, ν) is

$\mathbf{alx+bmy+cnz=0}$, which is a plane.

Note 1. *This plane passes through the centre of the conicoid and is called* **diametral plane** *of a system of parallel chords parallel to the line*

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}.$$

Note 2. *Conversely, every central plane is a diametral plane corresponding to same direction.*

EXAMPLES IX (C)

Type I. Ex. 1. Find the equation to the plane which cuts $x^2+4y^2-5z^2=1$ in a conic whose centre is the point (2, 3, 4).

[*Punjab B. Sc. (Hons.), 1951*]

Sol. Here $x_1=2, y_1=3, z_1=4, a=1, b=4, c=-5$.

$\therefore$ The required equation of the plane is

$$2x+12y-20z=4+36-80, \qquad \text{(Art. 9·22)}$$

or,
$$2x+12y-20z=-40,$$

or
$$x+6y-10z+20=0.$$

Ex. 2. Find the centre of the conic given by the equations

$$2x-2y-5z+5=0, \quad 3x^2+2y^2-15z^2=4.$$

[*Punjab B.Sc. (Hons.), 1955*]

[**Ans.** $(-2, 3, -1)$.]

Ex. 3. Prove that the centre of the section of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

by the plane through the three extremities of the different axes is the centroid of the triangle formed by those extremities.

Type II. Ex. 1. Find the locus of the centres of the sections of $ax^2+by^2+cz^2=1$ which touch $\alpha x^2+\beta y^2+\gamma z^2=1$.

(*Punjab Hons., 1958 ; Raj., 1952*)

Sol. Let (λ, μ, ν) be the centre of one of the sections of the conicoid

$$ax^2+by^2+cz^2=1 \qquad \ldots(1)$$

$\therefore$ equation of the section is

$$(x-\lambda)a\lambda+(y-\mu)\,b\mu+(z-\nu)c\nu=0 \qquad \ldots(2)$$

If (2) touches the conicoid $\alpha x^2+\beta y^2+\gamma z^2=1$,

then
$$a\lambda^2+b\mu^2+c\nu^2=\sqrt{\frac{a^2\lambda^2}{\alpha}+\frac{b^2\mu^2}{\beta}+\frac{c^2\nu^2}{\gamma}}.$$

$\therefore$ locus of (λ, μ, ν) is

$$(ax^2+by^2+cz^2)^2=\frac{a^2x^2}{\alpha}+\frac{b^2y^2}{\beta}+\frac{c^2z^3}{\gamma}.$$

Ex. 2. Show that the centres of sections of a conicoid that pass through a given line lie on a conic. (*Baroda, 1953*)

Ex. 3. Prove that the locus of the centres of parallel plane sections of a conicoid is a diameter. (*Punjab Hons., 1950*)

Ex. 4. Find the locus of the centres of sections of a conicoid that are at a constant distance from the centre. (*Allahabad 1963*)

[**Ans.** If the equation of the conicoid be $ax^2+by^2+cz^2=1$, and d is the constant distance of the section from the centre, the required locus is $(ax^2+by^2+cz^2)^2=d^2(a^2x^2+b^2y^2+c^2z^2)$.]

Type III. Ex. 1. Prove that the middle points of chords of $ax^2+by^2+cz^2=1$ which are parallel to $x=0$ and touch $x^2+y^2+z^2=r^2$ lie on the surface $by^2(bx^2+by^2+cz^2-br^2)+cz^2(cx^2+by^2+cz^2-cr^2)=0$.

Sol. Equations of any chord parallel to $x=0$ are

$$\frac{x-\alpha}{0}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad \ldots(1)$$

where (α, β, γ) are the coordinates of the middle point of this chord.

$\because$ (α, β, γ) lies on the diametral plane,

$\therefore$ $b\beta m+c\gamma n=0$ $\qquad \ldots(2)$

$\because$ (1) touches the sphere $x^2+y^2+z^2=r^2$,

$\therefore$ the perpendicular from (0, 0, 0) on (1) $=\pm r$,

or, $$\left[(\alpha^2+\beta^2+\gamma^2)-\left\{\frac{\beta m}{\sqrt{m^2+n^2}}+\frac{\gamma n}{\sqrt{m^2+n^2}}\right\}^2\right]^{\frac{1}{2}}=\pm r$$

or, $$(\alpha^2+\beta^2+\gamma^2)-\frac{(\beta m+\gamma n)^2}{m^2+n^2}=r^2,$$

or, $$(\alpha^2+\beta^2+\gamma^2-r^2)(m^2+n^2)=(\beta m+\gamma n)^2$$

or, $$\left(\beta+\frac{n}{m}\gamma\right)=(\alpha^2+\beta^2+\gamma^2-r^2)\left(1+\frac{n^2}{m^2}\right),$$

or, $$\left[\beta-\frac{b\beta}{c\gamma}\gamma\right]^2=(\alpha^2+\beta^2+\gamma^2-r^2)\left[1+\frac{b^2\beta^2}{c^2\gamma^2}\right]$$

$\therefore$ locus of (α, β, γ) is

$$\left[y-\frac{b}{c}y\right]^2=(x^2+y^2+z^2-r^2)\left[1+\frac{b^2y^2}{c^2z^2}\right],$$

or, $$(c-b)^2y^2z^2=(x^2+y^2+z^2-r^2)(b^2y^2+c^2z^2),$$

or, $$c^2y^2z^2+b^2y^2z^2-2bcy^2z^2=b^2y^2x^2+b^2y^4+b^2y^2z^2-b^2r^2y^2+c^2x^2z^2+c^2y^2z^2+c^2z^4-c^2r^2z^2,$$

or, $$by^2(bx^2+by^2+cz^2-br^2)+cz^2(cx^2+by^2+cz^2-cr^2)=0.$$

Ex. 2. Find the locus of the middle points of chords of the conicoid which pass through the point (α, β, γ). [**Ans.** $ax(x-\alpha)+by(y-\beta)+cz(z-\gamma)=0$.]

SECTION V

9·24. Enveloping cone : Def.

The locus of the tangent lines drawn from a given point to a given conicoid is a cone called the **enveloping cone** or **tangent cone** to the given conicoid having the given point as its vertex.

9·25. To find the equation of the enveloping cone of the conicoid $ax^2+by^3+cz^2=1$ with its vertex at $A(x_1, y_1, z_1)$.

The equation of the given conicoid is

$$ax^2+by^2+cz^2=1 \qquad \ldots(1)$$

Let A be the point (x_1, y_1, z_1).

Equations of **any** line through A are

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r, \text{ say.} \qquad ...(2)$$

Any point on (2) is (x_1+lr, y_1+mr, z_1+nr).

If it lies on (1), then

$$a(x_1+lr)^2+b(y_1+mr)^2+c(z_1+nr)^2=1,$$

or, $$r^2(al^2+bm^2+cn^2)+2r(alx_1+bmy_1+cnz_1)$$
$$+(ax_1^2+by_1^2+cz_1^2-1)=0 \qquad ...(3)$$

The line (2) will be a tangent line to the conicoid (1), if (2) meets (1) in two coincident points, *i.e.*, if (3) has equal roots, the condition for which is

$$4(alx_1+bmy_1+cnz_1)^2=4(al^2+bm^2+cn^2)(ax_1^2+by_1^2+cz_1^2-1) \qquad ...(4)$$

The locus of the tangent lines is obtained by eliminating l, m, n between (2) and (4).

$\therefore$ the required locus is

$$[ax_1(x-x_1)+by_1(y-y_1)+cz_1(z-z_1)]^2$$
$$=[a(x-x_1)^2+b(y-y_1)^2+c(z-z_1)^2](ax_1^2+by_1^2+cz_1^2-1)^* \qquad ...(5)$$

or, $$[a(x^2-2xx_1+x_1^2)+b(y^2-2yy_1+y_1^2)$$
$$+c(z^2-2zz_1+z_1^2)].(ax_1^2+by_1^2+cz_1^2-1)$$
$$=[(axx_1-ax_1^2+byy_1-by_1^2+czz_1-cz_1^2]^2,$$

or, $$[(ax^2+by^2+cz^2)+(ax_1^2+by_1^2+cz_1^2)$$
$$-2(axx_1+byy_1+czz_1)].(ax_1^2+by_1^2+cz_1^2-1)$$
$$=[axx_1+byy_1+czz_1)-(ax_1^2+by_1^2+cz_1^2)]^2,$$

or, $$[(ax^2+by^2+cz^2-1)+(ax_1^2+by_1^2+cz_1^2-1)$$
$$-2(axx_1+byy_1+czz_1-1)].(ax_1^2+by_1^2+cz_1^2-1)$$
$$=[(axx_1+byy_1+czz_1-1)-(ax_1^2+by_1^2+cz_1^2-1)]^2,$$

(Note this step)

---

* If the $S=ax^2+by^2+cz^2-1$, $S_1=ax_1^2+by_1^2+cz_1^2-1$,
$T=axx_1+byy_1+czz_1-1$,

Then (5) becomes $[(T+1)-(S_1+1)]^2=[(S+1)-2(T+1)+(S_1+1)]S_1$

or, $(T-S_1)^2=(S-2T+S_1)S_1$,

or, $T^2-2TS_1+S_1^2=SS_1-2TS_1+S_1^2$,

or, $SS_1=T^2$,

or, $(ax^2+by^2+cz^2-1)(ax_1^2+by_1^2+cz_1^2-1)=(axx_1+byy_1+czz_1-1)^2$.

or, $(ax^2+by^2+cz^2-1)(ax_1^2+by_1^2+cz_1^2-1)+(ax_1^2+by_1^2+cz_1^2-1)^2$
$-2(axx_1+byy_1+czz_1-1)(ax_1^2+by_1^2+cz_1^2-1)$
$=(axx_1+byy_1+czz_1-1)^2-2(axx_1+byy_1+czz_1-1)$,
$(ax_1^2+by_1^2+cz_1^2-1)+(ax_1^2+by_1^2+cz_1^2-1)^2$,

or, $$(\mathbf{ax^2+by^2+cz^2-1})(\mathbf{ax_1^2+by_1^2+cz_1^2-1})=(\mathbf{axx_1+byy_1+czz_1-1})^2,$$
which is the required equation of the enveloping cone.

Aliter. Let Q be any point on a tangent line drawn from A.

The coordinates of the point dividing AQ in the ratio of $k:1$ is
$$\left[\frac{kx+x_1}{k+1},\ \frac{ky+y_1}{k+1},\ \frac{kz+z_1}{k+1}\right].$$

If it lies on the given conicoid, then
$$a\left(\frac{kx+x_1}{k+1}\right)^2+b\left(\frac{ky+y_1}{k+1}\right)^2+c\left(\frac{kz+z_1}{k+1}\right)^2=1,$$

or, $$a(k^2x^2+x_1^2+2kxx_1)+b(k^2y^2+y_1^2+2kyy_1)+c(k^2z^2+z_1^2+2kzz_1)=k^2+2k+1,$$

or, $$k^2(ax^2+by^2+cz^2-1)+2k(axx_1+byy_1+czz_1-1)+(ax_1^2+by_1^2+cz_1^2-1)=0 \qquad \text{...(A)}$$

This is a quadratic equation in k.

$\because$ AQ touches the conicoid,

$\therefore$ (A) has equal roots, the condition for which is
$$4(axx_1+byy_1+czz_1-1)^2=4(ax^2+by^2+cz^2-1)(ax_1^2+by_1^2+cz_1^2-1).$$
or,
$$(\mathbf{ax^2+by^2+cz^2-1})(\mathbf{ax_1^2+by_1^2+cz_1^2-1})=(\mathbf{axx_1+byy_1+czz_1-1})^2,$$
which is the required equation of the enveloping cone.

Note. 1. *If* $S\equiv ax^2+by^2+cz^2-1$, $S_1\equiv ax_1^2+by_1^2+cz_1^2-1$ *and* $T\equiv axx_1+byy_1+czz_1-1$, *the above equation can be written as* $\mathbf{SS_1=T^2}$. *(Allahabad, 1950)*

Note 2. *The enveloping cone is satisfied by those points which satisfy both the equations* $ax^2+by^2+cz^2=1$ *and* $axx_1+byy_1+czz_1=1$. *The former is the equation of the conicoid and the later is the polar plane of the vertex* (x_1, y_1, z_1) *with respect to the given conicoid. Hence the enveloping cone may also be regarded as a cone with the given point* (x_1, y_1, z_1) *as the vertex and the guiding curve as the section of the conicoid by the polar plane of the vertex with respect to this conicoid.*

9·26. Enveloping cylinder : Def.

The locus of the tangent lines to a conicoid which are parallel to a given line, is a cylinder called the **enveloping cylinder** of the conicoid.

9·27. To find the equation of the enveloping cylinder of the conicoid $\mathbf{ax^2+by^2+cz^2=1}$, whose generators are parallel to the line $\mathbf{\frac{x}{l}=\frac{y}{m}=\frac{z}{n}}$.

The equation of the given conicoid is

$$ax^2+by^2+cz^2=1 \qquad ...(1)$$

Let (λ, μ, ν) be **any** point on a tangent line of (1) parallel to the given line

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \qquad ...(2)$$

The equations of this tangent line are

$$\frac{x-\lambda}{l}=\frac{y-\mu}{m}=\frac{z-\nu}{n}=r, \text{ say.} \qquad ...(3)$$

Any point on this line is $(\lambda+lr, \mu+mr, \nu+nr)$.

If it lies on (1). then

$$a(\lambda+lr)^2+b(\mu+mr)^2+c(\nu+nr)^2=1,$$

or,

$$r^2(al^2+bm^2+cn^2)+2r(al\lambda+bm\mu+cn\nu)+(a\lambda^2+b\mu^2+c\nu^2-1)=0 \qquad ...(4)$$

This is a quadratic equation in r.

$\because$ (3) touches (1),

$\therefore$ (4) has equal roots, the condition for which is

$$4(al\lambda+bm\mu+cn\nu)^2=4(al^2+bm^2+cn^2)(a\lambda^2+b\mu^2+c\nu^2-1).$$

$\therefore$ locus of (λ, μ, ν) is

$$\mathbf{(ax^2+by^2+cz^2-1)(al^2+bm^2+cn^2)=(alx+bmy+cnz)^2},$$

which is the required equation of the enveloping cylinder of the given conicoid.

Aliter. The enveloping cylinder whose generators are parallel to the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ is a limiting form of the enveloping cone whose vertex is (lr, mr, nr), when $r \longrightarrow \infty$.

The equation of the enveloping cone having vertex at (lr, mr, nr) is

$$(ax^2+by^2+cz^2-1)(al^2r^2+bm^2r^2+cn^2r^2-1)=(axlr+bymr+cznr-1)^2,$$

or,
$$(ax^2+by^2+cz^2-1)\left(al^2+bm^2+cn^2-\frac{1}{r^2}\right)=\left(axl+bmy+cnz-\frac{1}{r}\right)^2.$$

Taking the limits when $r\longrightarrow\infty$, we have

$$(\mathbf{ax^2+by^2+cz^2-1})(\mathbf{al^2+bm^2+cn^2})=(\mathbf{alx+bmy+cnz})^2,$$

which is the required equation of the enveloping cylinder.

EXAMPLES IX (D)

Type I. Ex. 1. Find the locus of points from which three mutually perpendicular tangent lines can be drawn to the surface $\mathbf{ax^2+by^2+cz^2=1}$. *(Jodhpur, 1962 ; I.A.S., 1955)*

Sol. Let (λ, μ, ν) be a point from which three mutually perpendicular tangent lines can be drawn to the surface $ax^2+by^2+cy^2=1$ ...(1)

The three mutually perpendicular tangent lines drawn from (λ, μ, ν) will be three mutually perpendicular generators of the enveloping cone of the conicoid (1) having vertex at (λ, μ, ν). The equation of the enveloping cone is

$$(ax^2+by^2+cz^2-1)(a\lambda^2+b\mu^2+c\nu^2-1)=(ax\lambda+by\mu+cz\nu-1)^2 \qquad ...(1)$$

$\because$ this cone has mutually perpendicular generators,

$\therefore$ sum of the caefficients of x^2, y^2, z^2 in (1) must be zero

$\therefore$ $a(b\mu^2+c\nu^2-1)+b(a\lambda^2+c\nu^2-1)+c(a\lambda^2+b\mu^2-1)=0,$

or $\Sigma a(b+c)\lambda^2=a+b+c.$

$\therefore$ locus of (λ, μ, ν) is $\Sigma a(b+c)x^2=a+b+c.$

Ex. 2. The section of the enveloping cone of the ellipsoid $\dfrac{\mathbf{x}^2}{\mathbf{a}^2}+\dfrac{\mathbf{y}^2}{\mathbf{b}^2}+\dfrac{\mathbf{z}^2}{\mathbf{c}^2}=1$, whose vertex is P, by the plane $\mathbf{z=0}$ is a rectangular hyperbola. Find the locus of P. *(Delhi, 1958 ; Bombay, 1960)*

Sol. The enveloping cone with vertex $P(\lambda, \mu, \nu)$ is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)\left(\frac{\lambda^2}{a^2}+\frac{\mu^2}{b^2}+\frac{\nu^2}{c^2}-1\right)=\left(\frac{\lambda x}{a^2}+\frac{\mu y}{b^2}+\frac{\nu z}{c^2}-1\right)^2$$

Its section by the plane $z=0$ is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{\lambda^2}{a^2}+\frac{\mu^2}{b^2}+\frac{\nu^2}{c^2}-1\right)=\left(\frac{\lambda x}{a^2}+\frac{\mu y}{b^2}\right)^2 \qquad ...(1)$$

If the section is a rectangular hyperbola, the sum of the coefficients of x^2 and y^2 in (1) is zero.

$$\therefore \quad \frac{1}{a^2}\left(\frac{\mu^2}{b^2}+\frac{\nu^2}{c^2}-1\right)+\frac{1}{b^2}\left(\frac{\lambda^2}{a^2}+\frac{\nu^2}{c^2}-1\right)=0,$$

or, $$\frac{1}{a^2b^2}(\lambda^2+\mu^2)+\frac{\nu^2}{c^2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)=\frac{1}{a^2}+\frac{1}{b^2}.$$

or $$\frac{\lambda^2+\mu^2}{a^2+b^2}+\frac{\nu^2}{c^2}=1.$$

$\therefore$ locus of P is $\dfrac{x^2+y^2}{a^2+b^2}+\dfrac{z^2}{c^2}=1.$

Ex. 3. Find the locus of a luminous point if the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

casts a circular shadow on the plane $z=0$. *(Bihar, 1961 ; Jodhpur, 1965)*

$$\left[\textbf{Ans.}\quad x=0,\ \frac{y^2}{b^2-a^2}+\frac{z^2}{c^2}=1;\ y=0,\ \frac{x^2}{a^2-b^2}+\frac{z^2}{c^2}=1.\right]$$

Ex. 4. The section of the enveloping cone of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

whose vertex is P by the plane $z=0$ is a parabola. Find the locus of P.

[**Ans.** $z=\pm c$.]

Type II. Ex. 1. Prove that the enveloping cylinders of the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$, whose generators are parallel to the line $\dfrac{x}{0}=\dfrac{y}{\pm\sqrt{a^2-b^2}}=\dfrac{z}{c}$, meet the plane $z=0$ in circles.

(Delhi Hons., 1954)

Sol. The enveloping cylinder of the given ellipsoid whose generators are parallel to the given line is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)\left(\frac{a^2-b^2}{b^2}+\frac{c^2}{c^2}\right)=\left(\pm\frac{\sqrt{a^2-b^2}}{b^2}\,y+\frac{cz}{c^2}\right)^2 \quad \text{...(Art. 9·27)}$$

It meets the plane $z=0$ in

$$z=0,\ \frac{a^2}{b^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)=\frac{a^2-b^2}{b^4}\,y^2,$$

or, $x^2+y^2=a^2$, $z=0$, which is a circle.

Ex. 2. Enveloping cylinders of the conicoid $ax^2+by^2+cz^2=1$ meet the plane $z=0$ in rectangular hyperbolas, show that the central perpendiculars to their planes of contact generate the cone

$$c(b^2x^2+a^2y^2)+ab(a+b)z^2=0.$$

Ex. 3. Show that the enveloping cylinder of the conicoid

$$ax^2+by^2cz+^2=1$$

with generators perpendicular to the z-axis meet the plane $z=0$ in a parabola.

SECTION VI

NORMALS TO A CONICOID

9.28. Normal : Def.

A normal at a point on a conicoid is the line drawn through that point perpendicular to the tangent plane at that point.

9.29. To find the equations of the normal at the point $(\mathbf{x_1}, \mathbf{y_1}, \mathbf{z_1})$ on the ellipsoid $\dfrac{\mathbf{x}^2}{\mathbf{a}^2}+\dfrac{\mathbf{y}^2}{\mathbf{b}^2}+\dfrac{\mathbf{z}^2}{\mathbf{c}^2}=\mathbf{1}$.

The equation of the given ellipsoid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad \ldots(1)$$

[In terms of direction ratios.]

The equation of the tangent plane at (x_1, y_1, z_1) is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=1 \qquad \ldots(2)$$

The direction ratios of the normal at (x_1, y_1, z_1) are

$$\frac{x_1}{a^2}, \frac{y_1}{b^2}, \frac{z_1}{c^2}.$$

$\therefore$ the equations of the normal at (x_1, y_1, z_1) are

$$\frac{\mathbf{x}-\mathbf{x}_1}{x_1/\mathbf{a}^2}=\frac{\mathbf{y}-\mathbf{y}_1}{\mathbf{y}_1/\mathbf{b}^2}=\frac{\mathbf{z}-\mathbf{z}_1}{\mathbf{z}_1/\mathbf{c}^2} \qquad \ldots(3)$$

[In terms of direction cosines.]

Let p be the length of the perpendicular from the centre $(0, 0, 0)$ on (2).

$$\therefore \quad p=1\Big/\sqrt{\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}+\frac{z_1^2}{c^4}}.$$

The direction ratios of the normal are $\dfrac{x_1}{a^2}, \dfrac{y_1}{b^2}, \dfrac{z_1}{c^2}$.

$\therefore$ direction cosines of the normal are

$$\frac{x_1}{a^2}\Big/\frac{1}{p}, \frac{y_1}{b^2}\Big/\frac{1}{p}, \frac{z_1}{c^2}\Big/\frac{1}{p},$$

or, $$\frac{px_1}{a^2}, \frac{py_1}{b^2}, \frac{pz_1}{c^2}.$$

$\therefore$ equations of the normal at (x_1, y_1, z_1) are

$$\frac{\mathbf{x}-\mathbf{x}_1}{\mathbf{p}\mathbf{x}_1/\mathbf{a}^2}=\frac{\mathbf{y}-\mathbf{y}_1}{\mathbf{p}\mathbf{y}_1/\mathbf{b}^2}=\frac{\mathbf{z}-\mathbf{z}_1}{\mathbf{p}\mathbf{z}_1/\mathbf{c}^2}.$$

Note. *The equations of the normal at* (x_1, y_1, z_1) *on the conicoid* $ax^2+by^2+cz^2=1$ *are*

$$\frac{x-x_1}{ax_1}=\frac{y-y_1}{by_1}=\frac{z-z_1}{cz_1}$$

or

$$\frac{x-x_1}{ax_1p}=\frac{y-y_1}{by_1p}=\frac{z-z_1}{cz_1p},$$

where p *is the length of the perpendicular from* $(0, 0, 0)$ *on the tangent plane at* (x_1, y_1, z_1).

9.30. Number of normals from a given point.

To show that from a given point (x_1, y_1, z_1), six normals, in general, can be drawn to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Proof. The given ellipsoid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1. \qquad ...(1)$$

The equations of the normal at (α, β, γ) on (1) are

$$\frac{x-\alpha}{\alpha/a^2}=\frac{y-\beta}{\beta/b^2}=\frac{z-\gamma}{\gamma/c^2} \qquad ...(2)$$

If it passes through the given point (x_1, y_1, z_1),

then
$$\frac{x_1-\alpha}{\alpha/a^2}=\frac{y_1-\beta}{\beta/b^2}=\frac{z_1-\gamma}{\gamma/c^2}=\lambda, \text{ say.}$$

$\therefore$
$$\alpha=\frac{a^2x_1}{a^2+\lambda},\ \beta=\frac{b^2y_1}{b^2+\lambda},\ \gamma=\frac{c^2z_1}{c^2+\lambda} \qquad ...(3)$$

$\because$ (α, β, γ) lies on (1),

$\therefore$
$$\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}=1,$$

or,
$$\frac{a^2x_1^2}{(a^2+\lambda)^2}+\frac{b^2y_1^2}{(b^2+\lambda)^2}+\frac{c^2z_1^2}{(c^2+\lambda)^2}=1 \qquad ...(4)$$

or,
$$a^2x_1^2(b^2+\lambda)^2(c^2+\lambda)^2+b^2y_1^2(a^2+\lambda)^2(c^2+\lambda)^2$$
$$+c^2z_1^2(a^2+\lambda)^2(b^2+\lambda)^2=(a^2+\lambda)^2(b^2+\lambda)^2(c^2+\lambda)^2$$

This is an equation of the sixth degree in λ, giving six values of λ, corresponding to which six points on the ellipsoid are obtained, the normals at each of which pass through the point (x_1, y_1, z_1).

Hence six normals, in general, can be drawn to the given ellipsoid from the given point.

This proves the proposition.

Note 1. *In case of the conicoid $ax^2+by^2+cz^2=1$, equation (4) becomes*

$$\frac{ax_1^2}{(a\lambda+1)^2}+\frac{by_1^2}{(b\lambda+1)^2}+\frac{cz_1^2}{(c\lambda+1)^2}=1,$$

and the relations (3) become

$$\alpha=\frac{x_1}{1+a\lambda},\ \beta=\frac{y_1}{1+b\lambda},\ \gamma=\frac{z_1}{1+c\lambda}.$$

Note 2. *All six normals need not be real.*

9·31. To show that the feet of the normals from (x_1, y_1, z_1) to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

lie on three cylinders which have a common curve of intersection.

Proof. The given ellipsoid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad \ldots(1)$$

The equations of the normal at (α, β, γ) on (1) are

$$\frac{x-\alpha}{\alpha/a^2}=\frac{y-\beta}{\beta/b^2}=\frac{z-\gamma}{\gamma/c^2} \qquad \ldots(2)$$

If it passes through the point (x_1, y_1, z_1), then

$$\frac{x_1-\alpha}{\alpha/a^2}=\frac{y_1-\beta}{\beta/b^2}=\frac{z_1-\gamma}{\gamma/c^2}=\lambda, \text{ say} \qquad \ldots(3)$$

From (3), the feet of the normals lie on

$$\frac{x_1-x}{x/a^2}=\frac{y_1-y}{y/b^2}=\frac{z_1-z}{z/c^2},$$

or,

$$\frac{a^2(x_1-x)}{x}=\frac{b^2(y_1-y)}{y}=\frac{c^2(z_1-z)}{z},$$

i.e., on the cylinders

$$b^2z(y_1-y)=c^2y(z_1-z),\ c^2x(z_1-z)=a^2z(x_1-x)$$

and

$$a^2y(x_1-x)=b^2x(y_1-y) \qquad \ldots(4)$$

From (3), we have

$$\alpha=\frac{a^2x_1}{a^2+\lambda},\ \beta=\frac{b^2y_1}{b^2+\lambda},\ \gamma=\frac{c^2z_1}{c^2+\lambda} \qquad \ldots(5)$$

From (5), the feet of the normals lie on the curve

$$x=\frac{a^2x_1}{a^2+\lambda},\ y=\frac{b^2y_1}{b^2+\lambda},\ z=\frac{c^2z_1}{c^2+\lambda} \qquad \ldots(6)$$

where λ is a parameter.

Hence the three cylinders given in (4) have a common curve of intersection given by (6).

9.32. Cubic curve through the feet of the six normals drawn from a point.

To prove that the feet of the normals from (x_1, y_1, z_1) to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

are the six points of intersection of the ellipsoid and a certain cubic curve.

Proof. The given ellipsoid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad \text{...(1)}$$

The six feet of the normals from (x_1, y_1, z_1) to (1) lie on the ellipsoid (1). ...(2)

Also, from (3) of Art. 9·29, the feet of the normals lie on the curve

$$x=\frac{a^2x_1}{a^2+\lambda},\quad y=\frac{b^2y_1}{b^2+\lambda},\quad z=\frac{c^2z_1}{c^2+\lambda} \qquad \text{...(3)}$$

where λ is a parameter.

We shall now show that (3) is a cubic curve. The curve (3) meets any plane

$$ux+vy+wz+d=0 \qquad \text{...(4)}$$

where $$u\left(\frac{a^2x_1}{a^2+\lambda}\right)+v\left(\frac{b^2y_1}{b^2+\lambda}\right)+w\left(\frac{c^2z_1}{c^2+\lambda}\right)+d=0,$$

or $$ua^2x_1(b^2+\lambda)(c^2+\lambda)+vb^2y_1(a^2+\lambda)(c^2+\lambda)$$
$$+wc^2z_1(a^2+\lambda)(b^2+\lambda)+d(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)=0,$$

which is cubic equation in λ, giving three values of λ.

Substituting these values of λ, one by one, in (3), we get three points in which the plane (4) meets the curve (3).

$\therefore$ the curve (3) is a cubic curve,

$\therefore$ the feet of the normals from (x_1, y_1, z_1) to (1) lie on the cubic curve (3). ...(5)

From (2) and (5), the feet of the normals from (x_1, y_1, z_1) to (1) are the six points of intersection of (1) and the cubic curve (3).

This proves the proposition.

9·33. To show that the six normals from (x_1, y_1, z_1) to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ lie on a cone of the second degree.

Proof. The given ellipsoid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad \ldots(1)$$

$\because$ the normal at (α, β, γ) passes through (x_1, y_1, z_1)

$$\therefore \qquad \alpha=\frac{a^2x_1}{a^2+\lambda}, \beta=\frac{b^2y_1}{b^2+\lambda}, \gamma=\frac{c^2z_1}{c^2+\lambda} \qquad \ldots(2)$$

Let the equations of a normal from (x_1, y_1, z_1) to (1) are

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \qquad \ldots(3)$$

Then $\qquad l=\frac{p\alpha}{a^2}=\frac{p}{a^2}\cdot\frac{a^2x_1}{a^2+\lambda}$, using (2)

$$=\frac{px_1}{a^2+\lambda}, \text{ or, } a^2+\lambda=\frac{px_1}{l} \qquad \ldots(4)$$

Similarly, $\qquad b^2+\lambda=\frac{py_1}{m} \qquad \ldots(5)$

and $\qquad c^2+\lambda=\frac{pz_1}{n} \qquad \ldots(6)$

Multiplying (4) by b^2-c^2, (5) by c^2-a^2 and (6) by a^2-b^2 and adding, we have

$$0=\frac{px_1}{l}(b^2-c^2)+\frac{py_1}{m}(c^2-a^2)+\frac{pz_1}{n}(a^2-b^2),$$

or, $\qquad \frac{x_1}{l}(b^2-c^2)+\frac{y_1}{m}(c^2-a^2)+\frac{z_1}{n}(a^2-b^2)=0 \qquad \ldots(7)$

(7) shows that (3) is a generator of the cone

$$\frac{x_1}{x-x_1}(b^2-c^2)+\frac{y_1}{y-y_1}(c^2-a^2)+\frac{z_1}{z-z_1}(a^2-b^2)=0,$$

which is a cone of the second degree.

$\therefore$ the six normals from (x_1, y_1, z_1) to (1) lie on a cone of the second degree.

This proves the proposition.

EXAMPLES IX (E)

Type I. Ex. 1. Prove that the normals to

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

at all points of its intersection with $lyz+mzx+nxy=0$ intersects the line

$$\frac{a^2x}{l(a^2-b^2)(c^2-a^2)}=\frac{b^2y}{m(b^2-c^2)(a^2-b^2)}=\frac{c^2z}{n(c^2-a^2)(b^2-c^2)}$$

(Raj., 1950)

Sol. The equations of the normal at (α, β, γ) is

$$\frac{x-\alpha}{\alpha/a^2}=\frac{y-\beta}{\beta/b^2}=\frac{z-\gamma}{\gamma/c^2} \qquad \ldots(1)$$

$\because$ (α, β, γ) lies on $lyz+mzx+nxy=0$,

$$\therefore \qquad l\beta\gamma+m\gamma\alpha+n\alpha\beta=0 \qquad \ldots(2)$$

$\therefore$ (1) intersect the given line if

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \frac{\alpha}{a^2} & \frac{\beta}{b^2} & \frac{\gamma}{c^2} \\ \frac{l}{a^2}(a^2-b^2)(c^2-a^2) & \frac{m}{b^2}(b^2-c^2)(a^2-b^2) & \frac{n}{c^2}(c^2-a^2)(b^2-c^2) \end{vmatrix}=0,$$

or if $\qquad \dfrac{1}{a^2b^2c^2(a^2-b^2)(b^2-c^2)(c^2-a^2)}$

$$\begin{vmatrix} \alpha a^2(b^2-c^2) & \beta b^2(c^2-a^2) & \gamma c^2(a^2-b^2) \\ \alpha(b^2-c^2) & \beta(c^2-a^2) & \gamma(a^2-b^2) \\ l(a^2-b^2)(b^2-c^2)(c^2-a^2) & m(a^2-b^2)(b^2-c^2)(c^2-a^2) & n(a^2-b^2)(b^2-c^2)(c^2-a^2) \end{vmatrix}=0$$

or if
$$\begin{vmatrix} \alpha a^2(b^2-c^2) & \beta b^2(c^2-a^2) & \gamma c^2(a^2-b^2) \\ \alpha(b^2-c^2) & \beta(c^2-a^2) & \gamma(a^2-b^2) \\ l & m & n \end{vmatrix}=0,$$

or if
$$\begin{vmatrix} a^2(b^2-c^2) & b^2(c^2-a^2) & c^2(a^2-b^2) \\ b^2-c^2 & (c^2-a^2) & (a^2-b^2) \\ \frac{l}{\alpha} & \frac{m}{\beta} & \frac{n}{\gamma} \end{vmatrix}=0,$$

on taking α, β, γ common from the first, second and third columns respectively,

or if
$$\begin{vmatrix} a^2(b^2-c^2) & b^2(c^2-a^2) & c^2(a^2-b^2) \\ b^2-c^2 & c^2-a^2 & a^2-b^2 \\ l\beta\gamma & m\gamma\alpha & n\alpha\beta \end{vmatrix}=0,$$

or if $\begin{vmatrix} 0 & b^2(c^2-a^2) & c^2(a^2-b^2) \\ 0 & c^2-a^2 & a^2-b^2 \\ 0 & m\gamma\alpha & n\alpha\beta \end{vmatrix}=0,$

on adding second and third columns to the first column and using (2) which is true. Hence the problem.

Ex. 2. Prove that the greatest value of shortest distance between the axis of x and a normal to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is $(b-c)$.

Ex. 3. Prove that the points on an ellipsoid the normals at which intersect a given straight line lie on the curve of intersection of the ellipsoid and a conicoid. *(Punjab, 1958)*

Ex. 4. The normals at P and Q points of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

meet the plane of XOY in A and B and make angles θ and ϕ with PQ, prove that PA $\cos\theta$+QB $\cos\phi=0$

Type II. Ex. 1. The normal at a variable point P on the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ meets the plane XOY in A and AQ is drawn parallel to OZ and equal to AP. Find the locus of Q.

[Agra, 1958 ; Punjab (Pakistan), 1956]

Sol. The equation of the normal at $P(x_1, y_1, z_1)$ is

$$\frac{x-x_1}{x_1/a^2}=\frac{y-y_1}{y_1/b^2}=\frac{z-z_1}{z_1/c^2} \qquad ...(1)$$

This meets the plane $z=0$ in A.

where $$z=0,\ \frac{x-x_1}{x_1/a^2}=\frac{y-y_1}{y_1/b^2}=\frac{-z_1}{z_1/c^2},$$

or, $$z=0,\ \frac{x-x_1}{x_1/a^2}=\frac{y-y_1}{y_1/b^2}=\frac{-z_1}{z_1/c^2}$$

$$=\frac{\sqrt{(x-x_1)^2+(y-y_1)^2+z_1^2}}{\sqrt{\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}+\frac{z_1^2}{c^4}}}=\frac{\text{PA}}{\sqrt{\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}+\frac{z_1^2}{c^4}}}$$

Now, the x and y coordinates of Q are the same as those of A, while the z-coordinate of Q is PA.

$\therefore$ For the point Q,

$$x=x_1-\frac{c^2x_1}{a^2}=\frac{x_1}{a^2}(a^2-c^2) \qquad ...(2)$$

$$y=y_1-\frac{c^2y_1}{b^2}=\frac{y_1}{b^2}(b^2-c^2) \qquad ...(3)$$

and $$z=-c^2\left(\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}+\frac{z_1^2}{c^4}\right)^{\frac{1}{2}} \qquad ...(4)$$

Now from (4),

$$\frac{z^2}{c^2}=c^2\left(\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}+\frac{z_1^2}{c^4}\right)=c^2\left(\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}\right)+\left(1-\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}\right),$$

[$\because$ (x_1, y_1, z_1) lies on the ellipsoid]

$$=1-\frac{x_1^2}{a^4}(a^2-c^2)-\frac{y_1^2}{b^4}(b^2-c^2)$$

$$=1-\frac{x^2}{a^2-c^2}-\frac{y^2}{b^2-c^2}, \text{ using (2) and (3).}$$

$\therefore$ The locus of Q is the ellipsoid

$$\frac{x^2}{a^2-c^2}+\frac{y^2}{b^2-c^2}+\frac{z^2}{c^2}=1.$$

Ex. 2. If Q be any point on the normal at P to the ellipsoid such that $3PQ=PG_1+PG_2+PG_3$, then prove that the locus of the point Q is

$$\frac{a^2x^2}{(2a^2-b^2-c^2)^2}=\frac{b^2y^2}{(2b^2-c^2-a^2)^2}=\frac{c^2z^2}{(2c^2-a^2-b^2)^2}=\frac{1}{9},$$

where G_1, G_2, G_3 are the points in which the normal at P meets the principal planes.

Ex. 3. If a length PQ be taken on the normal at any point P of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ such that $PQ=\frac{\lambda^2}{p^2}$, where λ is a constant and p is perpendicular from the origin to the tangent plane at P, the locus of Q is

$$\Sigma\frac{a^2x^2}{(a^2+\lambda^2)^2}=1.$$

Type III. Ex. 1. Prove that of the six normals from a point to an ellipsoid at least two are real.

Sol. From Art. 9·29, the sixth degree equation in λ giving six values of λ corresponding to the six feet of the normals drawn to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

from the point (x_1, y_1, z_1) is

$$(\lambda+a^2)^2(\lambda+b^2)^2(\lambda+c^2)^2-a^2x_1^2(\lambda+b^2)^2(\lambda+c^2)^2$$
$$-b^2y_1^2(\lambda+c^2)^2(\lambda+a^2)^2-c^2z_1^2(\lambda+a^2)^2(\lambda+b^2)^2=0 \qquad \ldots(1)$$

Let $F(\lambda)\equiv(\lambda+a^2)^2(\lambda+b^2)^2(\lambda+c^2)^2-a^2x_1^2(\lambda+b^2)^2(\lambda+c^2)^2$
$-b^2y_1^2(\lambda+c^2)^2(\lambda+a^2)^2-c^2z_1^2(\lambda+a^2)^2(\lambda+b^2)^2.$

Let us assume that $a^2>b^2>c^2$.

$\therefore$ $-a^2<-b^2<-c^2$.

Now $F(-\infty)=+\infty=+\text{ive}$,

$F(-a^2)=-a^2x_1^2(-a^2+b^2)^2(-a^2+c^2)^2=-\text{ive}$,

$F(-b^2)=-b^2y_1^2(-b^2+c^2)^2(-b^2+a^2)^2=-\text{ive}$,

$F(-c^2)=-c^2z_1^2(-c^2+a^2)^2(-c^2+b^2)^2=-\text{ive}$,

$F(+\infty)=+\infty=+\text{ive}$.

$\therefore$ From theory of equations, at least one real root of (1) lies between $-\infty$ and $-a^2$. and at least one real root of (1) lies between $-c^2$ and $+\infty$.

$\therefore$ at least two roots of (1) are real.

Hence at least two normals out of the six are real.

Ex. 2. (*i*) Prove that four normals to the ellipsoid $\frac{x^2}{a^2}+\frac{{}^2}{b^2}+\frac{z^2}{c^2}=1$ pass thsough any point of the curve of intersection of the ellipsoid and the conicoid $\sum x^2(b^2+c^2)=\sum b^2c^2$.

(*ii*) Prove that the four normals to the ellipsoid $\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{6}=1$ pass through any point of the curve of intersection of the ellipsoid and the conicoid $11x^2+10y^2+9z^2=74$. (*Sagar, 1961*

Type IV. Ex. 1. (i) Prove that the feet of the six normals from (α, β, γ) to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ lie on the curve of intersection of the ellipsoid and the cone $\sum \frac{a^2(b^2-c^2)\alpha}{x}=0$.

[*Agra, 1962 ; Delhi Hons., 1955 ; Punjab Hons, 1954 ; Raj., 1954 ; I.A.S., 1960 ; Raj. M.Sc. (Physics) 1963 ; Sagar 1960 ; Karnatak 1961*]

(ii) Show that the cubic curve through the feet of the six normals drawn from a point to an ellipsoid lies on the cone. (*Raj., 1961*)

Sol. (*i*) The equations of the normal at (x_1, y_1, z_1) of the given ellipsoid are $\frac{x-x_1}{x_1/a^2}=\frac{y-y_1}{y_1/b^2}=\frac{z-z_1}{z_1/c^2}$

$\because$ it passes through (α, β, γ),

$$\therefore \qquad \frac{\alpha-x_1}{x_1/a^2}=\frac{\beta-y_1}{y_1/b^2}=\frac{\gamma-z_1}{z_1/c^2}=\lambda, \text{ say.}$$

$\therefore$ the six feet of the normals from (α, β, γ) are given by

$$x_1=\frac{a^2\alpha}{a^2+\lambda}, \quad y_1=\frac{b^2\lambda}{b^2+\lambda}, \; z_1=\frac{c^2\lambda}{c^2+\lambda},$$

or,
$$\lambda=\frac{a^2\alpha}{x_1}-a^2 \qquad \ldots(2)$$

$$\lambda=\frac{b^2\beta}{y_1}-b^2 \qquad \ldots(3)$$

$$\lambda=\frac{c^2\gamma}{z_1}-c^2. \qquad \ldots(4)$$

Multiplying (2) by b^2-c^2, (3) by c^2-a^2 and (4) by a^2-b^2, and adding we have

$$0=\left(\frac{a^2\alpha}{x_1}-a^2\right)(b^2-c^2)+\left(\frac{b^2}{y_1}-b^2\right)(c^2-a^2)+\left(\frac{c^2\gamma}{z_1}-c^2\right)(a^2-b^2),$$

or
$$\sum \frac{a^2\alpha(b^2-c^2)}{x_1}=0.$$

$\therefore$ (x_1, y_1, z_1) lie on the cone

$$\Sigma \frac{a^2\alpha(b^2-c^2)}{x}=0 \qquad ...(5)$$

i.e., the feet of the normals lie on (5).

Also, the feet of the normals lie on the given ellipsoid.

Hence the feet of the six normals from (α, β, γ) to the ellipsoid lie on the curve of intersection of the ellipsoid and the cone (5).

Aliter.

From Art. 9·30, the feet of the normals drawn from the point (α, β, γ) to the given ellipsoid lie on the cylinders

$$(b^2-c^2)yz=b^2\beta z-c^2\gamma y \qquad ...(1)$$

$$(c^2-a^2)zx=c^2\gamma x-a^2\alpha z \qquad ...(2)$$

and
$$(a^2-b^2)xy=a^2\alpha y-b^2\beta x \qquad ...(3)$$

Multiplying (1) by $a^2\alpha$, (2) by $b^2\beta$ and (3) by $c^2\gamma$, and adding we have

$$\Sigma a^2(b^2-c^2)\alpha yz=0,$$

or,
$$\Sigma \frac{a^2(b^2-c^2)\alpha}{x}=0,$$

i.e., the feet of the six normal lie on this cone. Also they lie on the ellipsoid.

$\therefore$ they lie on the curve of intersection of this cone and the ellipsoid

(*ii*) The parametric equations of the cubic curve passing through the feet of the six normals drawn from the point (x_1, y_1, z_1) to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \text{ are}$$

$$x=\frac{a^2x_1}{a^2+\lambda},\ y=\frac{b^2y_1}{b^2+\lambda},\ z=\frac{c^2z_1}{c^2+\lambda}.$$

Substituting these values of x, y and z in the equatiou of the cone

$$\Sigma \frac{x_1(b^2-c^2)}{x-x_1}=0,$$

it is satisfied. Hence the cubic curve lies on this cone.

Ex. 2. Show that the six normals to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ drawn from the point $P(\alpha, \beta, \gamma)$ are the generators of a cone which has for its generators also the lines through P parallel to the axes and the perpendicular from P to its polar plane with respect to the ellipsoid.

If O be the centre of the conicoid, is PO a generator of the cone ?

(*Raj., 1959*)

Ex. 3. Prove that the generators of the cone which contains the normals from a given point to an ellipsoid are at right angles to their polars with respect to the ellipsoid. (*Raj., 1959 ; Sagar, 1962*)

Type V. Ex. 1. If P, Q, R; P′, Q′, R′ are the feet of the six normals from a point to the ellipsoid and the plane PQR is given by $lx+my+nz=p$, then the plane P′ Q′ R′ is given by

$$\frac{x}{a^2l}+\frac{y}{b^2m}+\frac{z}{c^2n}+\frac{1}{p}=0,$$

(I.A.S., 1961 ; Punjab, 1962 ; Raj , 1963 ; Allahabad, 1961 ; Agra, 1957 ; Bombay, 1959)

Sol. Let the equation of the plane ' 'be $l'x+m'y+n'z-p'=0$ ...(1)

The combined equation giving all the six feet of the normals from a given point (x_1, y_1, z_1) is $(lx+my+nz-p)(l'x+m'y+n'z-p')=0$ ...(2)

Let (α, β, γ) be a foot of the normal from the point (x_1, y_1, z_1).

$$\therefore \quad (l\alpha+m\beta+n\gamma-p)\,(l'\alpha+m'\beta+n'\gamma-p')=0 \qquad ...(3)$$

$\because$ the normal at (α, β, γ) passes through (x_1, y_1, z_1),

$$\therefore \quad \frac{x_1-\alpha}{\alpha/a^2}=\frac{y_1-\beta}{\beta/b^2}=\frac{z_1-\gamma}{\gamma/c^2}=\lambda, \text{ say.}$$

$$\therefore \quad \alpha=\frac{a^2x_1}{a^2+\lambda},\ \beta=\frac{b^2y_1}{b^2+\lambda},\ \gamma=\frac{c^2z_1}{c^2+\lambda} \qquad ...(4)$$

$\because$ (α, β, γ) lies on the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

$$\therefore \quad \frac{a^2x_1^2}{(a^2+\lambda)^2}+\frac{b^2y_1^2}{(b^2+\lambda)^2}+\frac{c^2z_1^2}{(c^2+\lambda)^2}=1 \qquad ...(5)$$

This equation, being a sixth degree equation in λ, gives six values of λ corresponding to the six feet of the normals.

Rewriting (3) after putting the values of α, β, γ from (4) we have

$$\left(\frac{a^2x_1l}{a^2+\lambda}+\frac{b^2y_1m}{b^2+\lambda}+\frac{c^2z_1n}{c^2+\lambda}-p\right)\left(\frac{a^2x_1l'}{a^2+\lambda}+\frac{b^2y_1m'}{b^2+\lambda}+\frac{c^2z_1n'}{c^2+\lambda}-p'\right)=0 \qquad ...(6)$$

$\because$ (5) and (6) are identical,

$\therefore$ comparing coefficients of like terms, we have

$$\frac{a^4ll'}{a^2}=\frac{b^4mm'}{b^2}=\frac{c^4nn'}{c^2}=-\frac{pp'}{1}$$

or

$$l'=-\frac{pp'}{a^2l},\ m'=-\frac{pp'}{b^2m},\ n'=-\frac{pp'}{c^2n}.$$

Substituting these values of l', m', n', in (1) we have

$$-\frac{pp'}{a^2l}x-\frac{pp'}{b^2m}y-\frac{pp'}{c^2n}z-p'=0.$$

or

$$\frac{x}{a^2l}+\frac{y}{b^2m}+\frac{z}{c^2n}+\frac{1}{p}=0.$$

Aliter. From Ex. 1 (*i*), type IV, the feet of the normals from (α, β, γ) to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1. \qquad ...(1)$$

lie on the cone $\Sigma \dfrac{a^2(b^2-c^2)\alpha}{x}=0,$

or,

$$\Sigma a^2(b^2-c^2)\alpha yz=0 \qquad ...(2)$$

Now, equation (2) does not contain terms of x^2, y^2 and z^2, and a pure constant term.

Let the equation of the plane P′Q′R′ be

$$l'x+m'y+n'z=p' \qquad ...(A)$$

$\therefore$ Combined equation of the planes PQR and P′Q′R′ is

$$(lx+my+nz-p)(l'x+m'y+n'z-p')=0 \qquad ...(3)$$

The equation of any surface passing through the points of intersection of (1) and (3) is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)+\lambda\,(lx+my+nz-p)\,(l'x+m'y+n'z-p')=0.$$

If it is the same as the equation of the cone through the feet P, Q, R ; P′, Q′, R′ of the six normals from (α, β, γ) to (1), then the coefficients of x^2, y^2 and z^2 are separately zero.

$$\therefore \qquad \frac{1}{a^2}+\lambda\, ll'=0, \quad \frac{1}{b^2}+\lambda\, mm'=0, \quad \frac{1}{c^2}+\lambda\, nn'=0,$$

and

$$-1+\lambda pp'=0,$$

or,

$$l'=-\frac{1}{a^2\lambda l}, \quad m'=-\frac{1}{b^2\lambda m}, \; n'=-\frac{1}{c^2\lambda n}, \quad p'=\frac{1}{\lambda p}.$$

Substituting these values of l', m', n', p' in (A), we have

$$-\frac{1}{\lambda}\left[\frac{x}{a^2l}+\frac{y}{b^2m}+\frac{z}{c^2n}\right]=\frac{1}{\lambda p}.$$

or,

$$\frac{x}{a^2l}+\frac{y}{b^2m}+\frac{z}{c^2n}+\frac{1}{p}=0.$$

Ex. 2. Two planes are drawn through the six feet of the normals drawn to the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ from a given point, each plane containing three feet ; prove that if A (α, β, γ) and A′$(\alpha', \beta', \gamma')$ be the poles of the planes with respect to the ellipsoid, then $\alpha\alpha'+a^2=\beta\beta'+b^2=\gamma\gamma'+c^2$,

and

$$AA'^2-OA^2-OA'^2=2(a^2+b^2+c^2).$$

[*Raj., 1961* ; *Bombay, 1959* (*second part*)]

Ex. 3. If the feet of three of the normals from P to the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ lie in the plane $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=1$, the feet of the other three lie in the plane $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}+1=0$, and P lies on the line $a(b^2-c^2)x=b(c^2-a^2)y=c(a^2-b^2)z.$

(*Gujarat, 1958* ; *Agra, 1954* ; *Banaras, 1950* ; *Vikram, 1962*)

SECTION VII

CONJUGATE DIAMETRAL PLANES AND CONJUGATE DIAMETERS

9·34. If P and Q be the points on the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ whose centre is O, and the diametral plane of OP passes through Q, to show that the diametral plane of OQ passes through P.

Proof. The given ellipsoid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ...(1)

Let P and Q be the points (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively.

The equations of OP are $\frac{x-0}{x_1-0}=\frac{y-0}{y_1-0}=\frac{z-0}{z_1-0}$,

or, $$\frac{x}{x_1}=\frac{y}{y_1}=\frac{z}{z_1}. \qquad ...(2)$$

The equations of the diametral plane of OP with respect to (1) is $$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=0^* \qquad ..(3)$$
(Art. 9·23)

If it passes through Q, then
$$\frac{x_2x_1}{a^2}+\frac{y_2y_1}{b^2}+\frac{z_2z_1}{c^2}=0,$$
or, $$\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}+\frac{z_1z_2}{c^2}=0 \qquad ...(4)$$

The symmetry of the result (4) shows that it is also the condition that the diametral plane of OQ should pass through P.

This proves the proposition.

Cor. Let P be a point on the ellipsoid, and Q a point on the diametral plane of OP and on the ellipsoid, and OR the line of intersection of the diametral planes of OP and Q, R being on the ellipsoid. Then the planes QOR, ROP and POQ are **conjugate diametral planes** and OP, OQ, OR are **conjugate semi-diameters.**

**The relation (3) shows that the diametral plane of any diameter is parallel to the tangent plane at the extremities of that diameter.*

9·36. To obtain the relations between the coordinates of the extremities of three conjugate semi-diameters of the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$.

Let OP, OQ and OR be three conjugate semi-diameters of the ellipsoid
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad \ldots(1)$$

Let the coordinates of P, Q and R are respectively
(x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3),

[A] $\because$ P, Q and R lie on (1),

$$\therefore \left.\begin{aligned}\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}&=1\\ \frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}+\frac{z_2^2}{c^2}&=1\\ \frac{x_3^2}{a^2}+\frac{y_3^2}{b^2}+\frac{z_3^2}{c^2}&=1\end{aligned}\right\} \qquad \ldots(\mathrm{I})$$

Also, since the diametral plane of any semi-diameter passes through the extremities of the other two conjugate semi-diameters (Art. 9·35),

$$\therefore \left.\begin{aligned}\frac{x_2x_1}{a^2}+\frac{y_2y_1}{b^2}+\frac{z_2z_1}{c^2}&=0\\ \frac{x_3x_1}{a^2}+\frac{y_3y_1}{b^2}+\frac{z_3z_1}{c^2}&=0\\ \frac{x_2x_3}{a^2}+\frac{y_2y_3}{b^2}+\frac{z_2z_3}{c^2}&=0\end{aligned}\right\} \qquad \ldots(\mathrm{II})$$

[B] By virtue of the relations (I), we see that
$$\frac{x_1}{a}, \frac{y_1}{b}, \frac{z_1}{c}; \frac{x_2}{a}, \frac{y_2}{b}, \frac{z_2}{c}; \frac{x_3}{a}, \frac{y_3}{b}, \frac{z_3}{c}$$
may be considered to be the direction cosines of some three lines.

Also, by virtue of the relations (II), these three lines are mutually at right angles.

We know that if l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3, be the direction cosines of three mutually perpendicular lines, then l_1, l_2, l_3; m_1, m_2, m_3; n_1, n_2, n_3 are also the direction cosines of three mutually perpendicular lines.

$$\therefore \quad \frac{x_1}{a}, \frac{x_2}{a}, \frac{x_3}{a}; \frac{y_1}{b}, \frac{y_2}{b}, \frac{y_3}{b} \text{ and } \frac{z_1}{c}, \frac{z_2}{c}, \frac{z_3}{c}$$

are the direction cosines of three mutually perpendicular lines

$\therefore$ we have the relations

$$\left.\begin{array}{l} \mathbf{x_1^2+x_2^2+x_3^2=a^2} \\ \mathbf{y_1^2+y_2^2+y_3^2=b^2} \\ \mathbf{z_1^2+z_2^2+z_3^2=c^2} \end{array}\right\} \ldots(\mathbf{III}) \qquad \left.\begin{array}{l} \mathbf{y_1z_1+y_2z_2+y_3z_3=0} \\ \mathbf{z_1x_1+z_2x_2+z_3x_3=0} \\ \mathbf{x_1y_1+x_2y_2+x_3y_3=0} \end{array}\right\} \ldots(\mathbf{IV})$$

[C] Solving first two equations of the relation (II), we have

$$\frac{x_1/a}{(y_2z_3-y_3z_2)/bc}=\frac{y_1/b}{(z_2x_3-z_3x_2)/ca}=\frac{z_1/c}{(x_2y_3-x_3y_2)/ab}$$

$$=\frac{\sqrt{\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}}}{\sqrt{\sum\left(\frac{y_2}{b}\cdot\frac{z_3}{c}-\frac{y_3}{b}\cdot\frac{z_2}{c}\right)^2}}=\pm\frac{1}{\sin 90^\circ}$$

$$=\pm 1,$$

since the angle between the lines whose direction cosines are

$$\frac{x_2}{a},\ \frac{y_2}{b},\ \frac{z_2}{c} \text{ and } \frac{x_3}{a},\ \frac{y_3}{b},\ \frac{z_3}{c} \text{ is } 90^\circ.$$

$$\left.\begin{array}{l} \text{Hence } \dfrac{\mathbf{x_1}}{\mathbf{a}}=\pm\left(\dfrac{\mathbf{y_2z_3-y_3z_2}}{\mathbf{bc}}\right),\ \dfrac{\mathbf{y_1}}{b}=\pm\dfrac{(\mathbf{z_2x_3-z_3x_2})}{\mathbf{ca}}, \\ \qquad \dfrac{\mathbf{z_1}}{\mathbf{c}}=\pm\dfrac{(\mathbf{x_2y_3-x_3y_2})}{\mathbf{ab}}. \\ \textbf{Similarly, } \dfrac{\mathbf{x_2}}{\mathbf{a}}=\pm\dfrac{(\mathbf{y_3z_1-z_3y_1})}{\mathbf{bc}},\ \textbf{etc.} \\ \text{and} \qquad \dfrac{\mathbf{x_3}}{\mathbf{a}}=\pm\dfrac{(\mathbf{y_1z_2-y_2z_1})}{\mathbf{bc}},\ \textbf{etc.} \end{array}\right\} \ldots(\mathbf{V})$$

[D]
$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}=x_1(y_2z_3-y_3z_2)+y_1(z_2x_3-z_3x_2)+z_1(x_2y_3-x_3y_2)$$

$$=\pm\left[\frac{x_1^2bc}{a}+\frac{y_1^2ca}{b}+\frac{z_1^2ab}{c}\right],\text{ using (V)}$$

$$=\pm abc\left[\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}\right]$$

$$=\pm\mathbf{abc} \qquad (\mathbf{VI})$$

9·37. Properties of conjugate semi-diameters.

(A) To prove that the sum of the squares of any three conjugate semi-diameters of an ellipsoid is constant.

Proof. Let $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$ and $R(x_3, y_3, z_3)$ be the extremities of three conjugate semi-diameters OP, OQ and OR of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$

Now, $\quad OP^2+OQ^2+OR^2=(x_1^2+y_1^2+z_1^2)+(x_2^2+y_2^2+z_2^2)+(x_3^2+y_3^2+z_3^2)$

$=(x_1^2+x_2^2+x_3^2)+(y_1^2+y_2^2+y_3^2)+(z_1^2+z_2^2+z_3^2)$

$=a^2+b^2+c^2$, using Art. 9·36, which is constant.

(B) To prove that the volume of the parallelopiped having three conjugate semi-diameters of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

as coterminous edges is constant.

Proof. Let $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$, $R(x_3, y_3, z_3)$ be the extremities of three conjugate semi-diameters OP, OQ and OR of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

From Mensuration, we have

The volume of the parallelopiped which has OP, OQ and OR for coterminous edges.

$=6$(volume of the tetrahedron O, PQR)

$$=6.\tfrac{1}{6}\begin{vmatrix} 0 & 0 & 0 & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix}=-\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \quad \ldots(1)$$

Now, $$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}\times\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$=\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}\times\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$$

$$=\begin{vmatrix} x_1^2+x_2^2+x_3^2 & x_1y_1+x_2y_2+x_3y_3 & x_1z_1+x_2z_2+x_3z_3 \\ x_1y_1+x_2y_2+x_3y_3 & y_1^2+y_2^2+y_3^2 & y_1z_1+y_2z_2+y_3z_3 \\ x_1z_1+x_2z_2+x_3z_3 & y_1z_1+y_2z_2+y_3z_3 & z_1^2+z_2^2+z_3^2 \end{vmatrix}$$

$$=\begin{vmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{vmatrix}, \text{ using Art. 9·36}$$

$=a^2b^2c^2,$ or, $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = \pm(abc).$

$\therefore$ (1) becomes,

the required volume$=-(\pm abc)=\pm abc.$

$\therefore$ volume in magnitude$=abc$, which is constant.

This proves the proposition.

(C) If OP, OQ, OR be three conjugate semi-diameters of the ellipsoid

$$\frac{\mathbf{x}^2}{\mathbf{a}^2}+\frac{\mathbf{y}^2}{\mathbf{b}^2}+\frac{\mathbf{z}^2}{\mathbf{c}^2}=1$$

and A_1, A_2, A_3 be the areas of the triangles QOR, ROP, POQ respectively, to prove that $A_1^2+A_2^2+A_3^2$ is constant.

Proof. Let $P(x_1, y_1, z_1)$ $Q(x_2, y_2, z_2)$, $R(x_3, y_3, z_3)$ be the extremities of conjugate semi-diameters OP, OQ, OR of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Let $l_r, m_r, n_r = 1, 2, 3$ be the direction cosines of the normals to the planes QOR, ROP, POQ respectively.

Projecting the triangle OQR on the yz-plane, *i.e.*, $x=0$, we get a triangle whose vertices are $(0, 0, 0)$, $(0, y_2, z_2)$ and $(0, y_3, z_3)$.

Its area is $\frac{1}{2}(y_2z_3-y_3z_2)$

or is $\pm\frac{bcx_1}{2a}$, using Art. 9·36.

But the area of this projection$=A_1l_1$.

$$\therefore \qquad A_1l_1=\pm bc.\frac{x_1}{2a}.$$

Similarly, $$A_1m_1=\pm ca.\frac{y_1}{2b}$$

and $$A_1n_1=\pm ab.\frac{z_1}{2c}.$$

$\therefore$ squaring and adding these relations we have

$$A_1^2(l_1^2+m_1^2+n_1^2)=\frac{b^2c^2}{4a^2}x_1^2+\frac{c^2a^2}{4b^2}y_1^2+\frac{a^2b^2}{4c^2}z_1^2$$

or $$A_1^2=\frac{b^2c^2}{4a^2}x_1^2+\frac{c^2a^2}{4b^2}y_1^2+\frac{a^2b^2}{4c^2}z_1^2 \qquad ...(1)$$

Similarly, $$A_2^2=\frac{b^2c^2}{4a^2}x_2^2+\frac{c^2a^2}{4b^2}y_2^2+\frac{a^2b^2}{4c^2}z_2^2 \qquad ...(2)$$

and $$A_3^2=\frac{b^2c^2}{4a^2}x_3^2+\frac{c^2a^2}{4b^2}y_3^2+\frac{a^2b^2}{4c^2}z_3^2 \qquad ...(3)$$

Adding (1), (2) and (3), we have

$$A_1^2+A_2^2+A_3^2=\frac{b^2c^2}{4a^2}(x_1^2+x_2^2+x_3^2)+\frac{c^2a^2}{4b^2}(y_1^2+y_2^2+y_3^2)$$
$$+\frac{a^2b^2}{4c^2}(z_1^2+z_2^2+z_3^2)$$
$$=\tfrac{1}{4}(b^2c^2+c^2a^2+a^2b^2), \text{ using Art. 9·36,}$$

which is constant.

This proves the proposition.

(D) To prove that the sum of squares of the projections of conjugate semi-diameters, (i) on a given line, (ii) on a given plane, is constant.

Proof. $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$, $R(x_3, y_3, z_3)$ be the extremities of conjugate semi-diameters OP, OQ, OR of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

(*i*) Let l, m, n be the direction cosines of the given line.

∴ projection on OP on this line

$$=(x_1-0)l+(y_1-0)m+(z_1-0)n=x_1l+y_1m+z_1n.$$

Similarly, the projections of OQ, and OR on this line are respectively

$$lx_2+my_2+nz_2 \text{ and } lx_3+my_3+nz_3.$$

∴ sum of the squares of the projections of OP, OQ and OR on the given line

$$=(lx_1+my_1+nz_1)^2+(lx_2+my_2+nz_2)^2+(lx_3+my_3+nz_3)^2$$
$$=l^2(x_1^2+x_2^2+x_3^2)+m^2(y_1^2+y_2^2+y_3^2)+n^2(z_1^2+z_2^2+z_3^2)$$
$$+2lm(x_1y_1+x_2y_2+x_3y_3)2mn+(y_1z_1+y_2z_2+y_3z_3)$$
$$+2nl(z_1x_1+z_2x_2+z_3x_3)$$
$$=a^2l^2+b^2m^2+c^2n^2, \text{ using Art. 9·36,}$$

which is constant.

(*ii*) Let l, m, n be the direction cosines of the normal to the given plane.

$\therefore$ square of the projection of OP on the given plane is
$$OP^2-(lx_1+my_1+nz_1)^2$$

Similarly, the squares of the projections of OQ and OR on the given plane are respectively
$$OQ^2-(lx_2+my_2+nz_2)^2,\ OR^2-(lx_3+my_3+nz_3)^2.$$

$\therefore$ sum of the squares of these projections
$$=OP^2+OQ^2+OR^2-(lx_1+my_1+nz_1)^2-(lx_2+my_2+nz_2)^2 -(lx_3+my_3+nz_3)^2$$
$=a^2+b^2+c^2-a^2l^2-b^2m^2-c^2n^2$, using Art. 9·36 and Art. 9·37 (A)
$=a^2(m^2+n^2)+b^2(n^2+l^2)+c^2(l^2+m^2)$, $(\because\ l^2+m^2+n^2=1)$
which is a constant.

This proves the proposition.

EXAMPLES IX (F)

Type I. Ex. 1. Find the equation of the plane through the extremities of three conjugate semi-diameters of an ellipsoid and show that it touches a fixed sphere. (*Lucknow Hons., 1962 ; Delhi Hons., 1961 ; I.A.S., 1953 ; Jodhpur, 1964 ; Agra, 1962*)

Sol. Let $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$ and $R(x_3, y_3, z_3)$ be the extremities of the conjugate semi-diameters OP, OQ and OR of the ellipsoid
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad ...(1)$$

Let the required plane PQR be
$$lx+my+nz=p \quad ...(2)$$

$\because$ (2) passes through P, Q, and R,

$\therefore$
$$lx_1+my_1+nz_1=p \quad ...(3)$$
$$lx_2+my_2+nz_2=p \quad ...(4)$$
and
$$lx_3+my_3+nz_3=p \quad ...(5)$$

Multiplying (3) by x_1, (4) by x_2 and (5) by x_3, and adding we have
$$l(x_1^2+x_2^2+x_3^2)+m(x_1y_1+x_2y_2+x_3y_3)+n(z_1x_1+z_2x_2+z_3x_3)$$
$$=p\ (x_1+x_2+x_3),$$
or,
$$la^2=p(x_1+x_2+x_3),\text{ using Art. 9·36}$$
or,
$$l=\frac{p}{a^2}\ (x_1+x_2+x_3).$$

Similarly, $m=\frac{p}{b^2}(y_1+y_2+y_3)$

and $n=\frac{p}{c^2}(z_1+z_2+z_3)$.

Substituting these values of l, m, n in (2), we have

$$p\left(\frac{x_1+x_2+x_3}{a^2}\right)x+p\left(\frac{y_1+y_2+y_3}{b^2}\right)y+p\left(\frac{z_1+z_2+z_3}{c^2}\right)z=p,$$

or, $\frac{\mathbf{x}}{\mathbf{a}^2}(\mathbf{x}_1+\mathbf{x}_2+\mathbf{x}_3)+\frac{\mathbf{y}}{\mathbf{b}^2}(\mathbf{y}_1+\mathbf{y}_2+\mathbf{y}_3)+\frac{\mathbf{z}}{\mathbf{c}^2}(\mathbf{z}_1+\mathbf{z}_2+\mathbf{z}_3)=\mathbf{1}$.

which is the required equation of the plane PQR.

Ex. 2. Show that the plane PQR touches the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{1}{3},$$

at the centroid of the triangle PQR. (*Punjab, 1959 ; Sagar, 1961 ; Aligarh, 1961 ; Karnatok, 1961 ; Calcutta Hons., 1962*)

Ex. 3. OP, OQ, OR are conjugate semi-diameters of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 ;$$

show that the sides of the triangle PQR touch the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{1}{2}.$$

Type II. Ex. 1. Prove that the locus of the foot of the perpendicular from the centre of the ellipsoid $\frac{\mathbf{x}^2}{\mathbf{a}^2}+\frac{\mathbf{y}^2}{\mathbf{b}^2}+\frac{\mathbf{z}^2}{\mathbf{c}^2}=\mathbf{1}$ on the plane PQR, through the extremities of conjugate semi-diameters OP, OQ and OR, is $\mathbf{a^2x^2+b^2y^2+c^2z^2=3(x^2+y^2+z^2)^2}$.

(*I.A.S., 1953 ; Punjab Hons., 1951 ; Raj., 1955*)

Sol. From Ex. 1, Type I above, the equation of the plane PQR is

$$\frac{x}{a^2}(x_1+x_2+x_3)+\frac{y}{b^2}(y_1+y_2+y_3)+\frac{z}{c^2}(z_1+z_2+z_3)=1. \quad ...(1)$$

Let (λ, μ, ν) e the foot of the perpendicular from the centre (0, 0, 0 on (1).

$\therefore$ equations of this perpendicular are

$$\frac{x}{\lambda}=\frac{y}{\mu}=\frac{z}{\nu} \quad ...(2)$$

$\because$ (2) is parallel to the normal of the plane (1),

$$\therefore \quad \frac{\lambda}{(x_1+x_2+x_3)/a^2}=\frac{\mu}{(y_1+y_2+y_3/b^2}=\frac{\nu}{(z_1+z_2+z_3)/c^2} \quad ...(3)$$

$\because$ (λ, μ, ν) lies on (1),

$$\therefore \quad \frac{\lambda}{a^2}(x_1+x_2+x_3)+\frac{\mu}{b^2}(y_1+y_2+y_3)+\frac{\nu}{c^2}(z_1+z_2+z_3)=1 \quad ...(4)$$

From (3), we have

$$\frac{a\lambda}{(x_1+x_2+x_3)/a}=\frac{b\mu}{(y_1+y_2+y_3)/b}=\frac{c\nu}{(z_1+z_2+z_3)/c}$$

$$=\frac{\sqrt{a^2\lambda^2+b^2\mu^2+c^2\nu^2}}{\sqrt{\frac{1}{a^2}(x_1+x_2+x_3)^2+\frac{1}{b^2}(y_1+y_2+y_3)^2+\frac{1}{c^2}(z_1+z_2+z_3)^2}}$$

$$=\frac{\sqrt{a^2\lambda^2+b^2\mu^2+c^2\nu^2}}{\sqrt{3}}, \text{ using Art. 9·36 (II)}$$

$\therefore$
$$\frac{x_1+x_2+x_3}{a^2}=\frac{\sqrt{3}\,\lambda}{\sqrt{a^2\lambda^2+b^2\mu^2+c^2\nu^2}},\quad \frac{y_1+y_2+y_3}{b^2}$$

$$=\frac{\sqrt{3}\,\mu}{\sqrt{a^2\lambda^2+b^2\mu^2+c^2\nu^2}}$$

and
$$\frac{z_1+z_2+z_3}{c^2}=\frac{\sqrt{3}\,\nu}{\sqrt{a^2\lambda^2+b^2\mu^2+c^2\nu^2}}.$$

Substituting these values in (4), we have

$$\sqrt{3}(\lambda^2+\mu^2+\nu^2)=\sqrt{a^2\lambda^2+b^2\mu^2+c^2\nu^2}$$

$\therefore$ locus of (λ, μ, ν) is

$$a^2x^2+b^2y^2+c^2z^2=3(x^2+y^2+z^2)^2.$$

Aliter. Let (λ, μ, ν) be the foot of the perpendicular from the centre on the plane PQR.

$\therefore$ equation of the plane PQR is

$$\lambda(x-\lambda)+\mu(y-\mu)+\nu(z-\nu)=0^*$$

or
$$x\lambda+y\mu+z\nu=\lambda^2+\mu^2+\nu^2. \qquad ...(1)$$

$\because$ it is the same as the equation of the plane PQR, *viz.*,

$$\frac{x}{a^2}(x_1+x_2+x_3)+\frac{y}{b^2}(y_1+y_2+y_3)+\frac{z}{c^2}(z_1+z_2+z_3)=1 \qquad ...(2)$$

$\therefore$ Comparing coefficients of like terms, we have

$$\frac{a^2\lambda}{x_1+x_2+x_3}=\frac{b^2\mu}{y_1+y_2+y_3}=\frac{c^2\nu}{z_1+z_2+z_3}=\frac{\lambda^2+\mu^2+\nu^2}{1}$$

$\therefore$
$$\frac{x_1+x_2+x_3}{a}=\frac{a\lambda}{\lambda^2+\mu^2+\nu^2} \qquad ...(3)$$

$$\frac{y_1+y_2+y_3}{b}=\frac{b\mu}{\lambda^2+\mu^2+\nu^2} \qquad ...(4)$$

and
$$\frac{z_1+z_2+z_3}{c}=\frac{c\nu}{\lambda^2+\mu^2+\nu^2} \qquad ...(3)$$

Squaring (3), (4) and (5), and then adding we have

$$\frac{(x_1+x_2+x_3)^2}{a^2}+\frac{(y_1+y_2+y_3)^2}{b^2}+\frac{(z_1+z_2+z_3)^2}{c^2}$$

$$=\frac{1}{(\lambda^2+\mu^2+\nu^2)^2}\,[a^2\lambda^2+b^2\mu^2+c^2\nu^2]$$

---

* $\because$ The plane PQR is a plane through (λ, μ, ν) and perpendicular to the line whose direction ratios are λ, μ, ν.

or $$3(\lambda^2+\mu^2+\nu^2)^2=a^2\lambda^2+b^2\mu^2+c^2\nu^2,$$ using Art. 9·36 (I) and (II).

$\therefore$ locus of (λ, μ, ν) is $\quad a^2x^2+b^2y^2+c^2z^2=3(x^2+y^2+z^2)^2$.

Ex. 2. Prove that the locus of the centre of the section of the ellipsoid

$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ by the plane PQR is the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{1}{3}.$$

Prove that this is also the locus of the centroid of the triangle PQR.

Ex. 3. Prove that the locus of the pole of the plane PQR is the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=3. \qquad (\textit{Sagar, 1961 ; Agra, 1954})$$

Type III. Ex. 1. (i) Find the locus of the equal conjugate diameters of the ellipsoid $\quad \dfrac{\mathbf{x}^2}{\mathbf{a}^2}+\dfrac{\mathbf{y}^2}{\mathbf{b}^2}+\dfrac{\mathbf{z}^2}{\mathbf{c}^2}=1.$

(*Allahabad, 1950, 1952, 1962 ; Delhi Hons., 1960, 1962 ; I.A.S., 1954 ; Jodhpur, 1963 ; Karnatak, 1961 ; Nagpur T.D.C., 1962 ; Raj., 1951, 1955, 1960 ; Agra, 1953, 1959 ; Vikram, 1961 ; Bombay, 1960*)

(ii) Show also that the plane through a pair of equal conjugate diameters touches the cone $\quad \Sigma\, \dfrac{\mathbf{x}^2}{\mathbf{a}^2(2\mathbf{a}^2-\mathbf{b}^2-\mathbf{c}^2)}=0.$

(*Agra, 1950 ; Karnatak, 1961 ; Raj., 1962*).

Sol. Let r be the length of each of the equal conjugate semi-diameter of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad \text{...(1)}$$

Let $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$ and $R(x_3, y_3, z_3)$ be the extremities of equi-conjugate semi-diameters OP, OQ and OR.

$$\therefore \quad r^2=x_1^2+y_1^2+z_1^2=x_2^2+y_2^2+z_2^2=x_3^2+y_3^2+z_3^2$$

$$\therefore \quad 3r^2=(x_1^2+x_2^2+x_3^2)+(y_1^2+y_2^2+y_3^2)+(z_1^2+z_2^2+z_3^2)$$

$$=a^2+b^2+c^2, \qquad \text{...(2)}$$

using Art. 9·36.

Let l, m, n be the direction cosines of the semi-diameter OP.

$\therefore$ Coordinates of P are (lr, mr, nr).

$\because$ P lies on (1),

$$\therefore \quad r^2\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)=1$$

or $$r^2\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)=l^2+m^2+n^2, \qquad \textbf{(Note this step)}$$

or, $$\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}=\frac{(l^2+m^2+n^2)}{r^2}=\frac{3(l^2+m^2+n^2)}{a^2+b^2+c^2} \qquad \text{...(3)}$$

using (2).

$\therefore$ locus of the equi-conjugate diameter of (1) is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{3(x^2+y^2+z^2)}{a^2+b^2+c^2}*$$

or $$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)(a^2+b^2+c^2)=3(x^2+y^2+z^2)$$

or $$\frac{x^2}{a^2}(2a^2-b^2-c^2)+\frac{y^2}{b^2}(2b^2-c^2-a^2)+\frac{z^2}{c^2}(2c^2-a^2-b^2)=0,$$

which is a cone.

(*ii*) The plane OQR through OQ and OR, is the diametral plane of OP.

$\therefore$ equation of the plane OQR is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=0 \qquad \text{..(A)}$$

This will touch the cone

$$\Sigma\frac{x^2}{a^2(2a^2-b^3-c^2)}=0 \text{ if}$$

$$\Sigma\frac{x_1^2}{a^4}.a^2(2a^2-b^2-c^2)=0, \qquad \text{(Art. 9 15)}$$

or if $$\Sigma x_1^2[3a^2-(a^2+b^2+c^2)]/a^2=0,$$

or if $$\Sigma 3x_1^2-\Sigma\frac{x_1^2}{a^2}(a^2+b^2+c^2)=0,$$

or if $$3(x_1^2+y_1^2+z_1^2)-(a^2+b^2+c^2)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}\right)=0$$

or if $3r^2=a^2+b^2+c^2$, $\because$ $\Sigma\frac{x_1^2}{a^2}=1$ and $x_1^2+y_1^2+z_1^2=r^2$,

which is true, since OP, OQ and OR are equal.

Aliter.

$$r^2=x_1^2+y_1^2+z_1^2=x_2^2+y_2^2+z_2^2=x_3^2+y_3^2+z_3^2$$

$$\therefore\ 3r^2=(x_1^2+x_2^2+x_3^2)+(y_1^2+y_2^2+y_3^2)+(z_1^2+z_2^2+z_3^2)$$

$$=a^2+b^2+c^2 \qquad \text{...(1)}$$

Equation of the plane QOR is the same as the equation of the diametral plane of OP.

$\therefore$ equation of the plane QOR is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=0 \qquad \text{...(2)}$$

The given cone may be written as

$$\frac{x^2}{a^2(r^2-a^2)}+\frac{y^2}{b^2(r^2-b^2)}+\frac{z^2}{c^2(r^2-c^2)}=0, \qquad \text{...(3)}$$

using (1).

---

* $\because$ the equations of OP are $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$.

$\therefore$ in view of (3), OP generates this cone.

Now, the plane QOR touches the cone (3), if the normal to the plane (2), *viz*.,

$$\frac{x}{x_1/a^2}=\frac{y}{y_1/b^2}=\frac{z}{z_1/c^2}$$

is a generator of the cone reciprocal to (3), which is obviously

$$a^2(r^2-a^2)x^2+b^2(r^2-b^2)y^2+c^2(r^2-c^2)z^2=0.$$

It is so if $(r^2-a^2)\ \dfrac{x_1^2}{a^2}+(r^2-b^2)\ \dfrac{y_1^2}{b^2}+(r^2-c^2)\ \dfrac{z_1^2}{c^2}=0,$

or if $$r^2\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}\right)-(x_1^2+y_1^2+z_1^2)=0.$$

or if $$r^2-(x_1^2+y_1^2+z_1^2)=0,$$

($\because$ P(x_1, y_1, z_1) lies on the given ellipsoid)

or if $$r^2-r^2=0,$$

or if $$0=0,$$

which is true.

Hence the problem.

Ex. 2. If λ, μ, ν be the angles between a set of equal conjugate diameters, prove that

$$\cos^2\lambda+\cos^2\mu+\cos^2\nu=\frac{3\Sigma(b^2-c^2)^2}{2(a^2+b^2+c^2)^2}$$

(*Raj., 1960 ; Agra, 1950 ; Aligarh, 1960*)

Ex 3. Show that any set of three equal conjugate diameters of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2+z^2}{b^2}=1$$

be on on a circular cone, and that the cosine of the angle between any two is $(a^2-b^2)/(a^2+2b^2)$.

Type IV. [**An important result. "If P be any point on an ellipsoid, and OQ, OR be any two conjugate semi-diameters of the ellipse in which the diametral plane of OP cuts the ellipsoid, then OP, OQ, OR are conjugate semi-diameters of the ellipsoid."**

Proof. $\because$ OQ and OR are conjugate semi-diameters of the ellipse in which the diametral plane of OP cuts the ellipsoid,

$\therefore$ OQ bisects all chords of the ellipse, parallel to OR ...(1)

Now these chords are also chords of the ellipsoid.

$\therefore$ the diametral plane of OR bisects these chords ...(2)

From (1) and (2), we conclude that the diametral plane of OR passes through OQ, *i.e.*, through Q.

Also, the diametral plane of OR passes through P.

$\therefore$ the diametral plane of OR is the plane POQ.

Similarly, the diametral plane of OQ is the plane ROP.

It is also given that the diametral plane of OP is the plane QOR.

$\therefore$ by Def., OP, OQ, OR are conjugate diameters of the ellipsoid. This proves the proposition].

Ex. 1. P is any point on the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ and 2α and 2β are the principal axes of the section of the ellipsoid by the diametral plane of OP, prove that

$$OP^2=a^2+b^2+c^2-\alpha^2-\beta^2, \quad \textbf{and that } \alpha\beta p=abc,$$

where p is the perpendicular from O to the tangent plane at P.

(Agra, 1956 ; I.A.S., 1952 ; Nagpur, T.D.C , 1960)

Sol. Let P be the point (x_1, y_1, z_1).

Let $Q(x_2, y_2, z_2)$ and $R(x_3, y_3, z_3)$ be the extremities of the sami-axes (which are conjugate diameters) of the ellipse which is the curve of intersection of the ellipsoid and the diametral plane of OP.

$\therefore$ by the above note, OP, OQ, and OR are conjugate semi-diameters of the ellipsoid.

Now, $$OQ^2=x_2^2+y_2^2+z_2^2=\alpha^2,$$
$$OR^2=x_3^2+y_3^2+z_3^2=\beta^2.$$

Also, $$OP^2=x_1^2+y_1^2+z_1^2.$$

$$\therefore \quad OP^2+\alpha^2+\beta^2=(x_1^2+x_2^2+x_3^2)+(y_1^2+y_2^2+y_3^2)+(z_1^2+z_2^2+z_3^2)$$
$$=a^2+b^2+c^2, \text{ using Art. 9·36.}$$
$$\therefore \quad OP^2=a^2+b^2+c^2-\alpha^2-\beta^2.$$

The tangent plane at P is $\dfrac{xx_1}{a^2}+\dfrac{yy_1}{b^2}+\dfrac{zz_1}{c^2}=1$

$$\therefore \quad \frac{1}{p^2}=\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}+\frac{z_1^2}{c^4}$$
$$=\frac{\Sigma(y_2z_3-y_3z_2)^2}{a^2b^2c^2}, \text{ using the relations}$$
$$\frac{x_1}{a}=\pm\left(\frac{y_2z_3-y_3z_2}{bc}\right), \text{ etc.}$$
$$\therefore \quad \frac{a^2b^2c^2}{p^2}=\Sigma(y_2z_3-y_3z_2)^2$$
$$=(x_2^2+y_2^2+z_2^2)(x_3^2+y_3^2+z_3^2)-(x_2x_3+y_2y_3+z_2z_3)^2,$$

using Lagrange's identity.

$$=\alpha^2\beta^2-(x_2x_3+y_2y_3+z_2z_3)^2 \qquad \ldots(1)$$

$\because$ **OQ and OR are perpendicular.**

$\therefore$ $\quad x_2x_3+y_2y_3+z_2z_3=0$

$\therefore$ **(1) becomes** $\quad \dfrac{a^2b^2c^2}{p^2}=\alpha^2\beta^2,$

or $$\alpha\beta \,.\, p=abc.$$

Aliter. Volume of the tetrahedran (P, OQR)

$$=\frac{1}{3}\text{ (Area of the } \triangle OQR) \times (\perp \text{ from P on the plane OQR)}$$

$$=\frac{1}{3}\left(\tfrac{1}{2}\alpha\beta\right)p=\frac{\alpha\beta p}{6} \qquad \ldots(1)$$

where p is the length of the perpendicular from P on the plane OQR, which is the same as the perpendicular from O on the tangent plane at P.

But the volume of the parallelopiped with coterminous edges OP, OQ and OR

$$=6(\text{volume of the tetrahedron P, OQR})$$

$$=6 \cdot \left(\frac{\alpha\beta p}{6}\right)=\alpha\beta p.$$

By Art. 9 37 (B), the volume of the parallelopiped with coterminous edges OP, OQ and OR$=abc$.

Hence $\qquad \alpha\beta p=abc.$

Ex. 2. If P, (x_1, y_1, z_1) is a point on the ellipsoid and (ξ_1, η_1, ζ_1), (ξ_2, η_2, ζ_2) are extremities of the principal axes of the section of the ellipsoid by the diametral plane of OP, prove that

$$\frac{\xi_1\xi_2}{a^2(b^2-c^2)}=\frac{\eta_1\eta_2}{b^2(c^2-a^2)}=\frac{\zeta_1\zeta_2}{c^2(a^2-b^2)},$$

$$(b^2-c^2)\frac{x_1}{\xi_1}+(c^2-a^2)\frac{y_1}{\eta_1}+(a^2-b^2)\frac{z_1}{\zeta_1}=0.$$

(I.A.S., 1951 ; Allahabad, 1960)

SECTION VIII

THE CONE

9·38. Standard equation of the cone.

A homogeneous equation of the form $ax^2+by^2+cz^2=0$ represents a standard equation of the cone whose vertex is the origin.

Note. *If* (x_1, y_1, z_1) *be any point on the cone* $ax^2+by^2+cz^2=0$, $(-x_1, -y_1, -z_1)$ *also lies on the cone.*

$\therefore$ *we can regard this cone a central conicoid whose centre is at the vertex of the cone. The coordinate planes are* **conjugate diametral planes** *and the coordinate axes are* **conjugate diameters.**

9·39. Important results.

As in the case of other central conicoids, the following results for the cone can also be established :

The equation of the **tangent plane** at the point (x_1, y_1, z_1) of the cone $ax^2+by^2+cz^2=0$ is $\mathbf{axx_1+byy_1+czz_1=0}$.

(*ii*) The plane $lx+my+nz=0$ touches the cone

$$ax^3+by^2+cz^2=0 \text{ if}$$

$$\frac{\mathbf{l^2}}{\mathbf{a}}+\frac{\mathbf{m^2}}{\mathbf{b}}+\frac{\mathbf{n^2}}{\mathbf{c}}=\mathbf{o}.$$

(*iii*) The equation of the **polar plane** of the point (x_1, y_1, z_1) with respect to the cone $ax^2+by^2+cz^2=0$ is $\mathbf{axx_1+byy_1+czz_1=o}$.

(*iv*) The equation of the plane of the section of the cone $ax^2+by^2+cz^2=0$, whose centre is (x_1, y_1, z_1), is $\mathbf{axx_1+byy_1+czz_1=ax_1{}^2+by_1{}^2+cz_1{}^2}$.

(*v*) The equation of the diametral plane of the cone

$$ax^2+by^2+cz^2=0,$$

which bisects the chords parallel to the line $\dfrac{x}{l}=\dfrac{y}{m}=\dfrac{z}{n}$, is

$$\mathbf{alx+bmy+cnz=o.}$$

(*vi*) The locus of the tangents drawn from the point $P(x_1, y_1, z_1)$ to the cone $ax^2+by^2+cz^2=0$ is the pair of the tangent planes

$$\mathbf{(ax^2+by^2+cz^2)(ax_1{}^2+by_1{}^2+cz_1{}^2)=(axx_1+byy_1+czz_1)^2}$$

whose line of intersection is OP, O being the vertex of the cone.

(*vii*) The diametral plane of OP is also the polar plane of P with respect to the cone.

EXAMPELS IX (G)

Type I. **Note. Normal plane : Def.** *The normal plane through any generator of a cone is the plane through that generator perpendicular to the tangent plane at any point of that generator.*

Ex. 1. Find the equation to the normal plane of the cone $\mathbf{ax^2+by^2+cz^2=0}$ which passes through the generator

$$\frac{\mathbf{x}}{\mathbf{l}}=\frac{\mathbf{y}}{\mathbf{m}}=\frac{\mathbf{z}}{\mathbf{n}}.$$

Sol. Equation of the given generator is

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}=r, \text{ say}$$

Any point on it is (lr, mr, nr).

The equation of the tangent plane at this point is

$$alx+bmy+cnz=0 \qquad ...(1)$$

Let the required normal plane be $l'x+m'y+n'z=0$ ...(2)

$\because$ it contains the generator, $\therefore$ $ll'+mm'+nn'=0$ ...(3)

$\because$ (2) is perpendicular to (1),

$\therefore$ $all'+bmm'+cnn'=0$ ...(4)

From (3) and (4),

$$\frac{ll'}{b-c}=\frac{mm'}{c-a}=\frac{nn'}{a-b}=\lambda, \text{ say.}$$

$$\therefore \quad l'=\frac{\lambda(b-c)}{l},\ m'=\frac{\lambda(c-a)}{m},\ n'=\frac{\lambda(a-b)}{n}.$$

$\therefore$ (2) becomes $\Sigma \dfrac{(b-c)x}{l}=0.$

Ex. 2. Lines drawn through the origin at right angles to normal planes of the cone $ax^2+by^2+cz^2=0$ generate the cone.

$$\Sigma \frac{a(b-c)^2}{x^2}=0. \qquad \textit{(Baroda, 1953 ; Raj., 1951)}$$

Type II. Ex. 1. Perpendicular tangent planes to $ax^2+by^2+cz^2=0$ intersect in generators of the cone $\Sigma a(b+c)x^2=0$.

(Allahabad, 1957 ; Punjab Hons., 1957 ; Bihar, 1962)

Sol. Let P be a point (α, β, γ).

$\therefore$ equations of OP are $\dfrac{x}{\alpha}=\dfrac{y}{\beta}=\dfrac{z}{\gamma}$ ...(1)

Equation of the tangent planes to $ax^2+by^2+cz^2=0$ is

$$(ax^2+by^2+cz^2)(a\alpha^2+b\beta^2+c\gamma^2)=(a\alpha x+b\beta y+c\gamma z)^2. \quad ...(2)$$

These tangent planes will be at right angles, if the sum of the coefficients of x^2, y^2 and z^2 in (2) is zero.

$$\therefore \quad (ab\beta^2+ac\gamma^2+ba\alpha^2+bc\gamma^2+ca\alpha^2+cb\beta^2=0$$

or $$a(b+c)\alpha^2+b(c+a)\beta^2+c(a+b)\gamma^2=0$$

$\therefore$ OP generates the cone $\Sigma\, a(b+c)x^2=0$.

Aliter.

The plane $lx+my+nz=0$ ...(1) touches the cone $ax^2+by^2+cz^2=0$

if $$\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=0,$$

or $$\Sigma bcl^2=0 \quad ...(2)$$

Eliminating n between (1) and (2), we have

$$l^2bc+m^2ca+\left(\frac{lx+my}{-z}\right)^2ab=0,$$

or $$l^2z^2bc+m^2z^2ca+l^2x^2ab+m^2y^2ab+2lmabxy=0,$$

or $$l^2(abx^2+bcz^2)+m^2(aby^2+caz^2)+2ablmxy=0 \quad ...(3)$$

This is a quadratic equation in $\dfrac{l}{m}$, which indicates the existence of two tangent planes.

$$\therefore \quad \frac{l_1l_2}{m_1m_2}=\frac{aby^2+caz^2}{abx^2+bcz^2},$$

where $\dfrac{l_1}{m_1}$ and $\dfrac{l_2}{m_2}$ are the roots of the quadratic (3) in $\dfrac{l}{m}$.

$\therefore \quad \dfrac{l_1 l_2}{aby^2+caz^2}=\dfrac{m_1 m_2}{abx^2+bcz^2}=\dfrac{n_1 n_2}{bcy^2+cax^2}$ (by Symmetry)

$\because$ the tangent planes are perpendicular,

$$\because \quad l_1 l_2+m_1 m_2+n_1 n_2=0$$

or $$aby^2+caz^2+abx^2+bcz^2+bcy^2+cax^2=0$$

or $\Sigma\, a(b+c)x^2=0$, where (x, y, z) are the coordinates of any point on the line of intersection.

Ex. 2. The lines of intersection of pairs of tangent planes to $ax^2+by^2+cz^2=0$ which touch along perpendicular generators lie on the cone

$$\Sigma a^2(b+c)x^2=0.$$ *(Agra, 1955)*

Ex. 3. Planes which cut $ax^2+by^2+cz^2=0$ in perpendicular generators touch

$$\Sigma \frac{x^2}{b+c}=0.$$

MISCELLANEOUS REVISION EXAMPLES ON CHAPTER IX

1. *If the line of intersection of two perpendicular tangent planes to the ellipsoid whose equation, referred to rectangnlar axes, is*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

passes through the fixed point $(0, 0, k)$, show that it lies on the cone

$$x^2(b^2+c^2-k^2)+y^2(c^2+a^2-k^2)+(z-k)^2(a^2+b^2)=0.$$

(Karnatak, 1961 ; Punjab Hons., 1960 ; Agra, 1963)

[**Hint.** Let the equations of the line of intersectian of two perpendicular tangent planes, through $(0, 0, k)$ be $\dfrac{x}{l}=\dfrac{y}{m}=\dfrac{z-k}{n}$ ...(1)

Let the equation of a tangent plane through (1) be $Ax+By+C(z-k)=0$,

or $$Ax+By+Cz=Ck \quad ...(2)$$

where $$Al+Bm+Cn=0. \quad ...(3)$$

$\because$ (2) touches the ellipsoid,

$$\therefore \quad A^2a^2+B^2b^2+C^2c^2=C^2k^2,$$

or $$A^2a^2+B^2b^2+C^2(c^2-k^2)=0 \quad ...(4)$$

$\because$ two tangent planes are at right angles,

$\therefore$ their normals are also at right angles.

$\therefore$ the lines whose direction ratios are given by (3) and (4), are at right angles.

$$\therefore \quad l^2(b^2+c^2-k^2)+m^2(c^2-k^2+a^2)+n^2(a^2+b^2)=0^* \quad ...(5)$$

Eliminate l, m, n between (1) and (5)].

Note. **The condition, that the two lines whose direction cosines l, m, n are given by $ul+vm+wn=0$, $al^2+bm^2+cn^2=0$, may be perpendicular, is*

$$\Sigma u^2(b+c)=0.$$

2. OP, OQ, OR are three radii of an ellipsoid whose centre is O and which are mutually at right angles to one another. Show that the plane PQR touches a sphere. *(Lucknow Hons., 1962 ; Raj., 1954)*

3. If the feet of the six normals from (α, β, γ) are (x_r, y_r, z_r), $r=1, 2, \ldots, 6$, prove that

$$a^2\alpha\Sigma\left(\frac{1}{x_r}\right)+b^2\beta\Sigma\left(\frac{1}{y_r}\right)+c^2\gamma\Sigma\left(\frac{1}{z_r}\right) \quad 0$$

(Punjab Hons., 1952 ; Allahabad, 1953 ; Bombay, 1961)

[**Hint**. *The feet of the normals from* (α, β, γ) *are given by*

$$\frac{a^2\alpha^2}{(a^2+\lambda)^2}+\frac{b^2\beta^2}{(b^2+\lambda)^2}+\frac{c^2\gamma^2}{(c^2+\lambda)^2}=1,$$

or $$\Sigma a^2\alpha^2(b^2+\lambda)^2(c^2+\lambda)^2=(a^2+\lambda)^2(b^2+\lambda)^2(c^2+\lambda)^2,$$

or $$\Sigma\lambda^6+2\lambda^5(a^2+b^2+c^2)+\ldots\ldots=0.$$

Let $\lambda_1, \lambda_2, \ldots\ldots \lambda_6$ *be its roots.*

$$\therefore \quad \Sigma\lambda_1=-2(a^2+b^2+c^2) \quad \ldots(1)$$

Also, the six feet of the normals are given by

$$x_r=\frac{a^2\alpha}{a^2+\lambda_r},\ y_r=\frac{b^2\beta}{b^2+\lambda_r},\ z_r=\frac{c^2\gamma}{c^2+\lambda_r},\ r=1, 2, \ldots, 6.$$

$$\therefore \quad a^2\alpha\Sigma\left(\frac{1}{x_r}\right)=a^2\alpha\Sigma\left(\frac{a^2+\lambda_r}{a^2\alpha}\right)=\Sigma(a^2+\lambda_r)=6a^2+\Sigma\lambda_1$$

$$b^2\beta\Sigma\left(\frac{1}{y_r}\right)=b^2\beta\Sigma\left(\frac{b^2+\lambda_r}{b^2\beta}\right)=\Sigma(b^2+\lambda_r)=6b^2+\Sigma\lambda_1$$

and $$c^2\gamma\Sigma\left(\frac{1}{z_r}\right)=c^2\gamma\Sigma\left(\frac{c^2+\lambda_r}{c^2\gamma}\right)=\Sigma(c^2+\lambda_r)=6c^2+\Sigma\lambda_1.$$

Add, etc.]

4. Through a fixed point $(k, 0, 0)$ pairs of perpendicular tangent lines are drawn to the surface $ax^2+by^2+cz^2=1$. Show that the plane through any pair touches the cone

$$\frac{(x-k)^2}{(ak^2-1)(b+c)}+\frac{y^2}{c(ak^2-1)-a}+\frac{z^2}{b(ak^2-1)-a}=0.$$

(I.A.S., 1956)

[**Hint.** The enveloping cone is $[a(x-k)^2+by^2+cz^2]$, $(ak^2-1)=a^2k^2(x-k)^2$.

Transferring the origin to $(k, 0, 0)$, it becomes

$$(ax^2+by^2+cz^2)(ak^2-1)=a^2k^2x^2,$$

or, $$ax^2-b(ak^2-1)y^2-c(ak^2-1)z^2=0.$$

Let the plane $ux+vy+wz=0$ through a pair cut the cone in perpendicular lines.

$$\therefore \quad [a-(b+c)(ak^2-1)]\,(u^2+v^2+w^2)=au^2-b(ak^2-1)v^2-c(ak^2-1)w^2,$$

or, $$[(b+c)(ak^2-1)]u^2+[c(ak^2-1)-a]v^2+[b(ak^2-1)-a]w^2=0.$$

$\therefore$ the normal to the plane generates the cone

$$Lx^2+My^2+Nz^2=0, \quad \text{where } L\equiv(b+c)(ak^2-1),$$
$$M\equiv c(ak^2-1)-a, \; N\equiv b(ak^2-1)-a.$$

$\therefore$ the plane touches the reciprocal cone which is

$$MNx^2+NLy^2+LMz^2=0,$$

or,
$$\frac{x^2}{L}+\frac{y^2}{M}+\frac{z^2}{N}=0.$$

$\therefore$ transforming origin back to $(k, 0, 0)$,

the equation is $\left[\dfrac{(x-k)^2}{L}+\dfrac{y^2}{M}+\dfrac{z^2}{N}=0, \text{etc.}\right]$

5. If OP, OQ, OR are conjugate diameters, and $p_1, p_2, p_3 ; \pi_1, \pi_2, \pi_3$ are their projections on any two given lines, then $p_1\pi_1+p_2\pi_2+p_3\pi_3$ is constant. *(Agra, 1956).*

[**Hint.** Let the lines be

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}, \quad \frac{x-\alpha'}{l'}=\frac{y-\beta'}{m'}=\frac{z-\gamma'}{n'} \text{ be two given lines.}$$

$\therefore$ $p_1\pi_1=(lx_1+my_1+nz_1)(l'x_1+m'y_1+n'z_1)$, write similar expressions for $p_2\pi_2, p_3\pi_3$ and add.]

6. If through any given point (α, β, γ) perpendiculars are drawn to any three conjugate diameters of the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$, the plane through the feet of the perpendiculars passes through the fixed point

$$\left(\frac{a^2\alpha}{a^2+b^2+c^2}, \frac{b^2\beta}{a^2+b^2+c^2}, \frac{c^2\gamma}{a^2+b^2+c^2}\right). \quad \textit{(Punjab, 1958)}$$

Sol. Let OP, OQ and OR be three conjugate semi-diameters of the given ellipsoid. Let the coordinates of P, Q and R be (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) respectively.

Let A be the foot of the perpendicular from $D(\alpha, \beta, \gamma)$ to OP.

$\therefore$ Coordinates of A are $(\lambda x_1, \lambda y_1, \lambda z_1)$,

$\because$ DA is perpendicular to OP,

$\therefore$ $x_1(\lambda x_1-\alpha)+y_1(\lambda y_1-\beta)+z_1(\lambda z_1-\gamma)=0$,

or $\lambda=(x_1\alpha+y_1\beta+z_1\gamma)/(x_1^2+y_1^2+z_1^2)$.

$\therefore$ coordinates of A become

$$\left[x_1 \cdot \frac{\Sigma x_1\alpha}{\Sigma x_1^2}, y_1\frac{\Sigma x_1\alpha}{\Sigma x_1^2}, z_1\frac{\Sigma x_1\alpha}{\Sigma x_1^2}\right].$$

Similarly, the coordinates of the feet B and C of the perpendiculars from (α, β, γ) on OQ and OR respectively are

$$\left(x_2\frac{\Sigma x_2\alpha}{\Sigma x_2^2}, y_2\frac{\Sigma x_2\alpha}{\Sigma x_2^2}, z_2\frac{\Sigma x_2\alpha}{\Sigma x_2^2}\right) \text{ and}$$

$$\left(x_3\frac{\Sigma x_3\alpha}{\Sigma x_3^2}, y_3\frac{\Sigma x_3\alpha}{\Sigma x_3^2}, z_3\frac{\Sigma x_3\alpha}{\Sigma x_3^2}\right).$$

$\therefore$ equation of the plane ABC is

$$\begin{vmatrix} x & y & z & 1 \\ x_1\Sigma x_1\alpha/\Sigma x_1^2 & y_1\dfrac{\Sigma x_1\alpha}{\Sigma x_1^2} & \dfrac{z_1\Sigma x_1\alpha}{\Sigma x_1^2} & 1 \\ x_2\Sigma x_2\alpha/\Sigma x_2^2 & y_2\Sigma x_2\alpha/\Sigma x_2^2 & \dfrac{z_2\Sigma x_2\alpha}{\Sigma x_2^2} & 1 \\ x_3\Sigma x_3\alpha/\Sigma x_3^2 & y_3\Sigma x_3\alpha/\Sigma x_3^2 & \dfrac{z_3\Sigma x_3\alpha}{\Sigma x_3^2} & 1 \end{vmatrix}=0,$$

or
$$\begin{vmatrix} x & y & z & 1 \\ x_1\Sigma x_1\alpha & y_1\Sigma x_1\alpha & z_1\Sigma x_1\alpha & \Sigma x_1^2 \\ x_2\Sigma x_2\alpha & y_2\Sigma x_2\alpha & z_2\Sigma x_2\alpha & \Sigma x_2^2 \\ x_3\Sigma x_3\alpha & y_3\Sigma x_3\alpha & z_3\Sigma x_3\alpha & \Sigma x_3^2 \end{vmatrix}=0$$

or
$$\begin{vmatrix} x & y & z & 1 \\ a^2\alpha & b^2\beta & c^2\gamma & a^2+b^2+c^2 \\ x_2\Sigma x_2\alpha & y_2\Sigma x_2\alpha & z_2\Sigma x_2\alpha & \Sigma x_2^2 \\ x_3\Sigma x_3\alpha & y_3\Sigma x_3\alpha & z_3\Sigma x_3\alpha & \Sigma x_3^2 \end{vmatrix}=0.$$

on adding the third and fourth rows to the second row and using Art. 9·36.

This plane clearly passes through the point

$$\left(\frac{a^2\alpha}{\Sigma a^2}, \frac{b^2\beta}{\Sigma a^2}, \frac{c^2\gamma}{\Sigma a^2}\right).$$

7. Prove that the locus of the point of intersection of three tangent planes to $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$, which are parallel to conjugate diametral planes of

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=1, \text{ is } \frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}.$$

(Punjab Hons., 1956)

What does this theorem become when $\alpha=\beta=\gamma$. *(Raj., 1950)*

[**Hint.** *OP, OQ, OR be any set of conjugate diameters of*

$\dfrac{x^2}{\alpha^2}+\dfrac{y^2}{\beta^2}+\dfrac{z^2}{\gamma^2}=1.$ *Let P, Q, R be* (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3).

$\therefore$ *three conjugate diametral planes are*

$$\frac{xx_r}{\alpha^2}+\frac{yy_r}{\beta^2}+\frac{zz_r}{\gamma^2}=0, \quad r=1, 2, 3. \qquad \ldots(1)$$

Let three tangent planes to ellipsoid

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ *parallel to (1) be* $l_rx+m_ry+n_rz$

$=\sqrt{a^2l_r^2+b^2m_r^2+c^2n_r^2}$, $r=1, 2, 3.$

$\therefore$ *these are* $\frac{xx_r}{\alpha^2}+\frac{yy_r}{\beta^2}+\frac{zz_r}{\gamma^2}=\sqrt{\frac{a^2x_r^2}{\alpha^4}+\frac{b^2y_r^2}{\beta^4}+\frac{c^2y_r^2}{\gamma^4}}$,

$r=1, 2, 3.$

To obtain the required locus, square and add these equations.

When $\alpha=\beta=\gamma, \frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=1$

becomes $x^2+y^2+z^2=\alpha^2$,

whose conjugate diametral planes are mutually perpendicular and the locus is the director sphere $x^2+y^2+z^2=a^2+b^2+c^2$.]

8. OP, OQ and OR are conjugate diameters of an ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

At Q and R tangent lines are drawn parallel to OP, and p_1, p_2, are their distances from O. The perpendicular from O to the tangent plane at right angles to OP is p. Prove that

$$p^2+p_1^2+p_2^2=a^2+b^2+c^2.$$ *(Raj., 1952)*

[**Hint.** Let $OP=r_1$, $OQ=r_2$, $OR=r_3$.

Equations to OP are

$$\frac{x}{x_1/r_1}=\frac{y}{y_1/r_1}=\frac{z}{z_1/r_1}.$$

Equations of the line through Q parallel to OP are

$$\frac{x-x_2}{x_1/r_1}=\frac{y-y_2}{y_1/r_1}=\frac{z-z_2}{z_1/r_1} \qquad ...(1)$$

$\therefore$ p_1^2=square of the perpendicular from O on (1)

$$=(x_2^2+y_2^2+z_2^2)-\left(\frac{x_1}{r_1}x_2+\frac{y_1}{r_1}y_2+\frac{z_1}{r_1}z_2\right)^2$$

Similarly, $p_2^2=x_3^2+y_3^2+z_3^2-\left(\frac{x_1x_3}{r_1}+\frac{y_1y_3}{r_1}+\frac{z_1z_3}{r_1}\right)^2$

Equation of the plane perpendicular to OP is

$$\frac{xx_1}{r_1}+\frac{yy_1}{r_1}+\frac{zz_1}{r_1}=p.$$

$\because$ it is a tangent plane,

$$\therefore \qquad p^2=\frac{a^2x_1^2+b^2y_1^2+c^2z_1^2}{r_1^2},$$

$$\therefore \quad p^2+p_1^2+p_2^2=\frac{a^2x_1^2+b^2y_1^2+c^2z_1^2}{r_1^2}+x_2^2+y_2^2+z_2^2-\frac{(x_1x_2+y_1y_2+z_1z_2)^2}{r_1^2}$$

$$+x_3^2+y_3^2+z_3^2-\frac{(x_1x_3+y_1y_3+z_1z_3)^2}{r_1^2}, \text{ etc.}]$$

9. Prove that two normals to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

lie in the plane

$$lx+my+nz=0$$

and the line joining their feet has direction cosines proportional to

$$a^2(b^2-c^2)mn,\ b^2(c^2-a^2)nl,\ c^2(a^2-b^2)lm.$$

Also, obtain the coordinates of thes points. *(Punjab Hons., 1957)*

[**Hint.** *Equations of the normal at* (x_1, y_1, z_1) *are*

$$\frac{x-x_1}{x_1/a^2}=\frac{y-y_1}{y_1/b^2}=\frac{z-z_1}{z_1/c^2}.$$

If it lies in the plane

$$lx+my+nz=0,$$

$$l\frac{x_1}{a^2}+m\frac{y_1}{b^2}+n\frac{z_1}{c^2}=0$$

and

$$lx_1+my_1+nz_1=0$$

Solving for x_1, y_1, z_1, *we have*

$$\frac{x_1}{a^2mn(b^2-c^2)}=\frac{y_1}{b^2nl\,(c^2-a^2)}=\frac{z_1}{c^2lm(a^2-b^2)}$$

$$\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}\right)^{\frac{1}{2}}\Big/\sqrt{\Sigma a^2m^2n^2(b^2-c^2)^2}$$

$$=\pm 1/\sqrt{\Sigma a^2m^2n^2(b^2-c^2)^2} \qquad ...(1)$$

This gives the coordinates of the required points. Find the direction ratios of the join of these two points.]

Index

[The numbers refer to the pages]